Optimized Parameters of Optimized Schwarz Waveform Relaxation Methods for The Heat Equation

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INTERNAL REPORT

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Chapter 1

Introduction

The Schwarz domain decomposition methods is a procedure to parallelize and solve partial differential equations numerically, where each iteration involves the solutions of the original equations on smaller subdomains. It was original proposed by H. A. Schwarz [7] in 1870 as a technique to prove the existence of a solution to the Laplace equation on a domain which is a combination of a rectangle and a circle. The idea was then used by P. L. Lions [4], [5], [6] as parallel algorithms in solving partial differential equations. Since then, many kind of domain decomposition methods have been developped, to improve the performance of the classical domain decomposition method. One of the main streams in this direction is to replace the Dirichlet transmission condition by Robin and Ventcell transmission conditions and then calculate the convergence rates. Using different transmissions condition gives different convergence rates and we need to optimize to get the best transmission conditions, the methods are then called the optimized Schwarz methods. In [1] and [2], D. Bennequin, M. Gander and L. Halpern show that the problem of optimizing the convergence rates is in fact a new class of best approximation problems and suggest a new method to solve this class of problems. The authors consider the model problem of optimizing the convergence factors for advection-diffusion equations. In this report, we use their methods to check the results announced in [3] and then extend the results to optimized Robin and Ventcell transmission conditions for 2 dimensional heat equations.

Chapter 2

Optimized Schwarz Waveform Relaxation Methods For The One Dimensional Heat Equation

2.1 Optimized Schwarz Waveform Relaxation Methods For The One Dimensional Heat Equation With Robin Transmission Condition

In this section, we consider the optimized Schwarz waveform relaxation method. The algorithm is

$$\begin{cases} (\partial_t - \nu\partial_{xx})u_1^k = f & \text{in } \Omega_1 \times (0,T), \\ u_1^k(x,0) = u_0(x) & \text{in } \Omega_1, \\ (\partial_x + \frac{p}{2\nu})u_1^k(L,.) = (\partial_x + \frac{p}{2\nu})u_2^{k-1}(L,.) & \text{in } (0,T), \end{cases} \tag{2.1.1}$$

$$\begin{cases} (\partial_t - \nu\partial_{xx})u_2^k = f & \text{in } \Omega_2 \times (0,T), \\ u_2^k(x,0) = u_0(x) & \text{in } \Omega_2, \\ (\partial_x - \frac{p}{2\nu})u_2^k(0,.) = (\partial_x - \frac{p}{2\nu})u_1^{k-1}(0,.) & \text{in } (0,T). \end{cases}$$

Let h_L and h_0 be given in $oH^{\frac{3}{4}}(0,T)$. Let (e_1, e_2) be the solution in $H^{3,\frac{3}{2}}(\Omega_1 \times (0,T)) \times H^{3,\frac{3}{2}}(\Omega_2 \times (0,T))$ of the problem

$$\begin{cases} (\partial_t - \nu\partial_{xx})e_1 = 0 & \text{in } \Omega_1 \times (0,T), \\ e_1(x,0) = 0 & \text{in } \Omega_1, \\ (\partial_x + \frac{p}{2\nu})e_1(L,.) = h_L & \text{in } (0,T), \end{cases} \tag{2.1.2}$$

$$\begin{cases} (\partial_t - \nu\partial_{xx})e_2 = 0 & \text{in } \Omega_2 \times (0,T), \\ e_2(x,0) = 0 & \text{in } \Omega_2, \\ (\partial_x - \frac{p}{2\nu})e_2(0,.) = h_0 & \text{in } (0,T). \end{cases}$$

We have

$$\mathfrak{F}e_1(x,\omega) = \frac{2\nu}{\sqrt{4i\omega\nu} + p}\mathfrak{F}h_L \exp(\sqrt{\frac{i\omega}{\nu}}(x-L)), \tag{2.1.3}$$

and

$$\mathfrak{F}e_2(x,\omega) = -\frac{2\nu}{\sqrt{4i\omega\nu} + p}\mathfrak{F}h_0 \exp(-\sqrt{\frac{i\omega}{\nu}}x). \tag{2.1.4}$$

We have also

$$\mathfrak{F}(\mathfrak{g}_D(h_L,h_0)) = \mathfrak{F}((\partial_x e_2 + \frac{p}{2\nu}e_2)(L,.), (\partial_x e_1 - \frac{p}{2\nu}e_1)(0,.)). \tag{2.1.5}$$

We have

$$\begin{aligned} \partial_x\mathfrak{F}e_2(L,\omega) + \frac{p}{2\nu}\mathfrak{F}e_2(L,\omega) &= -\frac{2\nu}{2\sqrt{i\omega\nu}+p}(-\sqrt{\frac{i\omega}{\nu}})\exp(-\sqrt{\frac{i\omega}{\nu}}L)\mathfrak{F}h_0 \\ &\quad -\frac{2\nu}{2\sqrt{i\omega\nu}+p}\frac{p}{2\nu}\exp(-\sqrt{\frac{i\omega}{\nu}}L)\mathfrak{F}h_0 \\ &= \frac{2\sqrt{i\omega\nu}-p}{2\sqrt{i\omega\nu}+p}\exp(-\sqrt{\frac{i\omega}{\nu}}L)\mathfrak{F}h_0, \end{aligned} \tag{2.1.6}$$

and

$$\begin{aligned} \partial_x\mathfrak{F}e_1(0,\omega) - \frac{p}{2\nu}\mathfrak{F}e_1(0,\omega) &= \frac{2\nu}{2\sqrt{i\omega\nu}+p}\sqrt{\frac{i\omega}{\nu}}\exp(-\sqrt{\frac{i\omega}{\nu}}L)\mathfrak{F}h_L \\ &\quad -\frac{2\nu}{2\sqrt{i\omega\nu}+p}\frac{p}{2\nu}\exp(-\sqrt{\frac{i\omega}{\nu}}L)\mathfrak{F}h_0 \\ &= \frac{2\sqrt{i\omega\nu}-p}{2\sqrt{i\omega\nu}+p}\exp(-\sqrt{\frac{i\omega}{\nu}}L)\mathfrak{F}h_L. \end{aligned} \tag{2.1.7}$$

From (2.1.5), (2.1.6), (2.1.7), we get

$$\mathfrak{F}(\mathfrak{g}_D(h_L,h_0)) = \frac{2\sqrt{i\omega\nu}-p}{2\sqrt{i\omega\nu}+p}\exp(-L\sqrt{\frac{i\omega}{\nu}})\mathfrak{F}(h_0,h_L). \tag{2.1.8}$$

Therefore

$$\mathfrak{F}(\mathfrak{g}_D^2(h_L,h_0)) = (\frac{2\sqrt{i\omega\nu}-p}{2\sqrt{i\omega\nu}+p})^2\exp(-2L\sqrt{\frac{i\omega}{\nu}})\mathfrak{F}(h_L,h_0). \tag{2.1.9}$$

Consequently

$$|\mathfrak{F}(\mathfrak{g}_D^2(h_L,h_0))| = |(\frac{2\sqrt{i\omega\nu}-p}{2\sqrt{i\omega\nu}+p})^2\exp(-2L\sqrt{\frac{i\omega}{\nu}})||\mathfrak{F}(h_L,h_0)| \tag{2.1.10}$$

$$= \exp(-L\sqrt{\tfrac{2|\omega|}{\nu}})\tfrac{(\sqrt{2|\omega|\nu}-p)^2+2|\omega|\nu}{(\sqrt{2|\omega|\nu}+p)^2+2|\omega|\nu}|\mathfrak{F}(h_L,h_0)|.$$

Thus for $k\in\mathbb{N}$,

$$|\mathfrak{F}\mathfrak{g}_D^{2k}(h_L,h_0)|(\omega) = \exp(-kL\sqrt{\frac{2|\omega|}{\nu}})(\frac{(\sqrt{2|\omega|\nu}-p)^2+2|\omega|\nu}{(\sqrt{2|\omega|\nu}+p)^2+2|\omega|\nu})^k|\mathfrak{F}(h_L,h_0)|(\omega).$$

Therefore

$$\begin{aligned}||\mathfrak{g}_D^{2k}h_L||_{H^{\frac{3}{4}}(\mathbb{R})} &= \int_{-\infty}^{+\infty}(1+|\omega|^2)^{\frac{3}{4}}|\mathfrak{F}\mathfrak{g}_D^{2k}h_L(\omega)|^2d\omega\\ &= \int_{-\infty}^{+\infty}(1+|\omega|^2)^{\frac{3}{4}}\exp(-2kL\sqrt{\frac{2|\omega|}{\nu}})\times\\ &\quad\times(\frac{(\sqrt{2|\omega|\nu}-p)^2+2|\omega|\nu}{(\sqrt{2|\omega|\nu}+p)^2+2|\omega|\nu})^{2k}|\mathfrak{F}h_L(\omega)|^2d\omega.\end{aligned}$$

Using the Lebesgue dominated convergence theorem with the notice that $(\frac{(\sqrt{2|\omega|\nu}-p)^2+2|\omega|\nu}{(\sqrt{2|\omega|\nu}+p)^2+2|\omega|\nu})^{2k}<1$, we can see that $\{||\mathfrak{g}_D^{2k}h_L||_{H^{\frac{3}{4}}(\mathbb{R})}\}$ converges to 0 when k tends to ∞. Similarly, $\{||\mathfrak{g}_D^{2k}h_0||_{H^{\frac{3}{4}}(\mathbb{R})}\}$ converges to 0 when k tends to ∞.

For $k\in\mathbb{N}$,

$$|\mathfrak{F}\mathfrak{g}_D^{2k}(h_L,h_0)|(\omega) = \exp(-kL\sqrt{\frac{2|\omega|}{\nu}})(\frac{(\sqrt{2|\omega|\nu}-p)^2+2|\omega|\nu}{(\sqrt{2|\omega|\nu}+p)^2+2|\omega|\nu})^k|\mathfrak{F}(h_L,h_0)|(\omega).$$

We define the convergence factor by

$$\rho(\omega; p, L) = \exp(-L\sqrt{2\frac{\omega}{\nu}})\frac{(\frac{p}{\nu} - \sqrt{2\frac{\omega}{\nu}})^2 + 2\frac{\omega}{\nu}}{(\frac{p}{\nu} + \sqrt{2\frac{\omega}{\nu}})^2 + 2\frac{\omega}{\nu}}. \tag{2.1.11}$$

Put $\frac{\omega}{\nu} = \bar{\omega}$, $\frac{p}{\nu} = \bar{p}$ and $\bar{\rho}(\bar{\omega}; \bar{p}, L) = \rho(\omega; p, L)$, we need to consider the following min-max problem

$$\min_{\bar{p}\in\mathbb{R}} \max_{\bar{\omega}\in[\bar{\omega}_{min}, \bar{\omega}_{max}]} \bar{\rho}(\bar{\omega}; \bar{p}, L), \tag{2.1.12}$$

where $\bar{\omega}_{min} = \frac{\pi}{2T\nu}$, $\bar{\omega}_{max} = \frac{\pi}{2\Delta t\nu}$. Since

$$\exp(-L\sqrt{2\bar{\omega}})\frac{(|\bar{p}| - \sqrt{2\bar{\omega}})^2 + 2\bar{\omega}}{(|\bar{p}| + \sqrt{2\bar{\omega}})^2 + 2\bar{\omega}} \leq \exp(-L\sqrt{2\bar{\omega}})\frac{(-|\bar{p}| - \sqrt{2\bar{\omega}})^2 + 2\bar{\omega}}{(-|\bar{p}| + \sqrt{2\bar{\omega}})^2 + 2\bar{\omega}},$$

so the minimum is attained for $\bar{p} \geq 0$, we only need to consider problem (2.1.12) in the case $\bar{p} \geq 0$, or the following problem

$$\min_{\bar{p}\geq 0} \max_{\bar{\omega}\in[\bar{\omega}_{min}, \bar{\omega}_{max}]} \bar{\rho}(\bar{\omega}; \bar{p}, L). \tag{2.1.13}$$

We have the following theorems for the overlapping case ($L > 0$)

Theorem 2.1.1. *We suppose that L is small and $\bar{\omega}_{max}$ is large.*

a) For $L(\bar{\omega}_{max})^{\frac{3}{4}}$ small (which means $L \backsim C(\bar{\omega}_{max})^{-\frac{3}{4}-\gamma}, \gamma > 0$), problem (2.1.13) *has a unique solution*

$$\bar{p}_* \backsim 2\sqrt{2}(\bar{\omega}_{min}\bar{\omega}_{max})^{\frac{1}{4}}$$

then

$$||\bar{\rho}(\bar{\omega}, \bar{p}_*, L)||_\infty \backsim 1 - 2\sqrt{2}(\frac{\bar{\omega}_{min}}{\bar{\omega}_{max}})^{\frac{1}{4}},$$

where the asymptotic expansion is based on the scale of $(\omega_{max})^{-1}$.

b) For $L(\bar{\omega}_{max})^{\frac{3}{4}}$ large (which means $L \backsim C(\bar{\omega}_{max})^{-\frac{3}{4}+\gamma}, \gamma > 0$) and $L < \nu\sqrt{\frac{10\sqrt{2}-12}{\bar{\omega}_{min}}}$, the problem (2.1.13) *has a unique solution*

$$\bar{p}_* \backsim (4\bar{\omega}_{min})^{\frac{1}{3}}(L)^{-\frac{1}{3}},$$

then

$$||\bar{\rho}(\omega, \bar{p}_*, L)||_\infty \backsim 1 - 4(\frac{\bar{\omega}_{min}}{2})^{\frac{1}{6}}(L)^{\frac{1}{3}},$$

where the asymptotic expansion is based on the scale of L.

Theorem 2.1.2.

For $L = C_1\Delta x$, $\Delta t = C_2\Delta x$, $\Delta x \le \min\big\{\big(\frac{(17-12\sqrt{2})2TC_2}{\pi^2(C_1\nu^{-1})^4}\big)^{\frac{1}{3}}, \frac{C_2^3}{\pi^2(C_1\nu^{-1})^4 2T}\big\}$, *then problem* (2.1.13) *has a unique solution*

$$\bar{p}_* \backsim \frac{2\sqrt{\pi}}{(2TC_2\nu^2)^{\frac{1}{4}}}\Delta x^{-\frac{1}{4}},$$

then

$$||\bar{\rho}(\bar{\omega}, \bar{p}_*, L)||_\infty \backsim 1 - \frac{2\sqrt{2}C_2^{\frac{1}{4}}}{2T^{\frac{1}{4}}}\Delta x^{\frac{1}{4}} + O(\Delta x^{\frac{1}{2}}).$$

For $L = C_1\Delta x$, $\Delta t = C_2\Delta x^2$, $\Delta x \le \min\big\{\big(\frac{\pi^2 2T(C_1\nu^{-1})^4}{4C_2^3}\big)^{\frac{1}{2}}, \big(\frac{(10\sqrt{2}-14)2T}{\pi(C_1\nu^{-1})^2}\big)^{\frac{1}{2}}\big\}$, *then problem* (2.1.13) *has a unique solution*

$$\bar{p}_* \backsim \big(\frac{4\pi}{2TC_1\nu}\big)^{\frac{1}{3}}\Delta x^{-\frac{1}{3}},$$

then

$$||\bar{\rho}(\bar{\omega}, \bar{p}_*, L)||_\infty \backsim 1 - 4\big(\frac{\pi C_1^2\nu^{-1}}{4T}\big)^{\frac{1}{6}}\Delta x^{\frac{1}{3}} + O(\Delta x^{\frac{2}{3}}).$$

Remark 2.1.1.

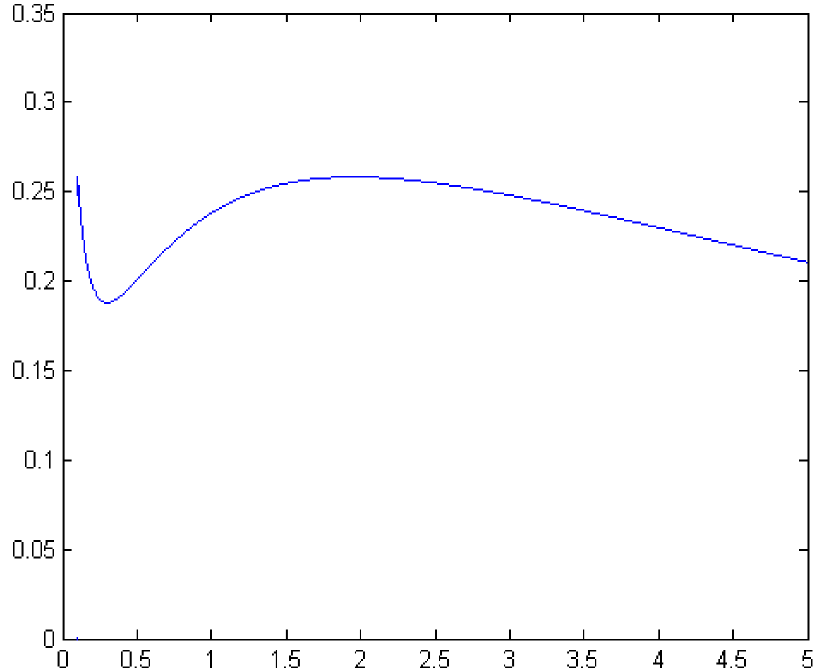


Figure 2.1.1

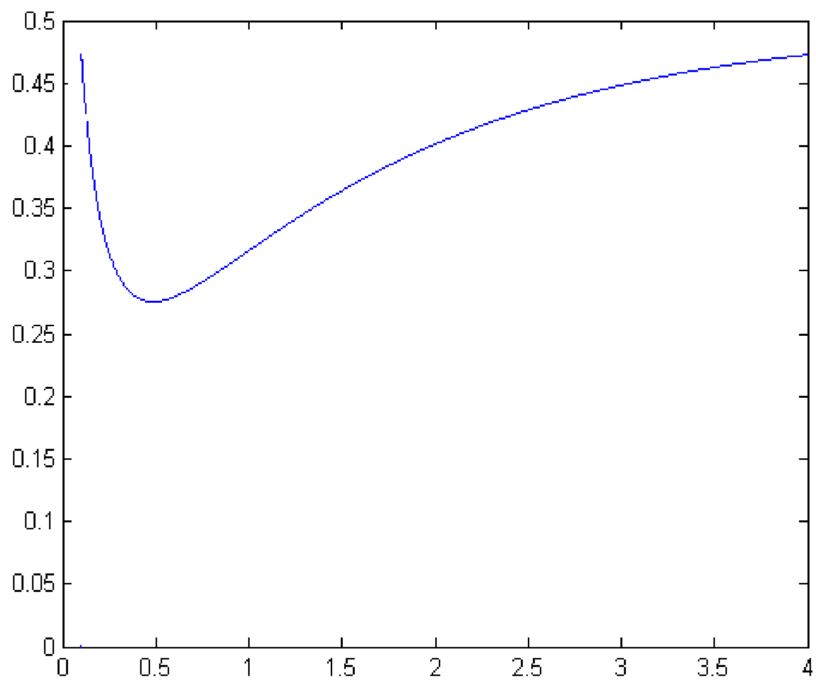


Figure 2.1.2

Figure 2.1 is the graph of $\bar{\rho}$ with respect to $\bar{\omega}$ for some $\bar{p}$. In the first cases of the previous two theorems, we can prove that the solution $\bar{p}_*$ of (2.1.12) *can be obtained by equilibrating the boundary on the right hand side and the maximal point of the graph. In the second cases $\bar{\omega}_2 > \bar{\omega}_{max}$, we equilibrate the two boundaries to get $\bar{p}_*$ (figure 2.2).*

For the non-overlapping case, we have the following result

Theorem 2.1.3. *Problem* (2.1.13) *has one and only one solution which is given by $\bar{\omega} = \bar{\omega}_{min}$ and $\bar{\omega} = \bar{\omega}_{max}$; $\bar{p} = 2\frac{\sqrt{\pi}}{(2T\nu^2)^{\frac{1}{4}}}\Delta t^{-\frac{1}{4}}$.*

$$\min_{\bar{p}} \max_{\bar{\omega}} \bar{\rho} \backsim 1 - (\frac{32C_1}{T})^{\frac{1}{4}} \Delta t^{\frac{1}{4}}.$$

2.1.1 Proof of the Theorems in the Overlapping Case

Putting

$$h_L(\bar{p}) = \max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega},\bar{p},L) = ||\rho(\bar{\omega};\bar{p},L)||_\infty,$$

we recall that $(\bar{p}^*, h_L(\bar{p}^*))$ is a strict local minimum of $h_L(\bar{p})$ if and only if there exists ϵ positive such that for all $\bar{p}$ in $(\bar{p}^*-\epsilon, \bar{p}^*+\epsilon)$, we have $h_L(\bar{p}) > h_L(\bar{p}^*)$.

In order to prove the theorems, we need the following lemma as in [1]

Lemma 2.1.1. *If $(\bar{p}^*, h_L(\bar{p}^*))$ is a strictly local minimum of $h_L(\bar{p})$, then it is the global minimum of $h_L(\bar{p})$ and $\bar{p}^*$ is the unique solution of* (2.1.13).

Proof of Lemma 2.1.1

We denote $\mathcal{D}(z_0,\delta) = \{z\in\mathbb{C}, |\frac{z-z_0}{z+z_0}| < \delta\}$, and $D^L_\delta = \{p|h_L(p)\leq\delta\}$.

We first prove that D^L_δ is a convex set. Let $\bar{p}_1$ and $\bar{p}_2$ be to elements of D^L_δ, we have that

$$||\exp(-L\sqrt{i\bar{\omega}})\frac{\frac{\bar{p}_1}{\nu}-2\sqrt{i\bar{\omega}}}{\bar{p}_1+2\sqrt{i\bar{\omega}}}||_\infty \leq \delta.$$

Thus $\forall\bar{\omega} \in [\bar{\omega}_{min},\bar{\omega}_{max}]$,

$$|\exp(-L\sqrt{i\bar{\omega}})\frac{\bar{p}_1-2\sqrt{i\bar{\omega}}}{\bar{p}_1+2\sqrt{i\bar{\omega}}}| \leq \delta.$$

Hence

$$\exp(-L\sqrt{\frac{\bar{\omega}}{2}})|\frac{\bar{p}_1-2\sqrt{i\bar{\omega}}}{\bar{p}_1+2\sqrt{i\bar{\omega}}}| \leq \delta.$$

Therefore

$$|\frac{\bar{p}_1-2\sqrt{i\bar{\omega}}}{\bar{p}_1+2\sqrt{i\bar{\omega}}}| \leq \delta\exp(L\sqrt{\frac{\bar{\omega}}{2}}).$$

This means $\bar{p}_1 \in \mathcal{D}(2\sqrt{i\bar{\omega}}, \delta\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$.

Similarly, we have also $\bar{p}_2 \in \mathcal{D}(2\sqrt{i\bar{\omega}}, \delta\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$.

According to Lemma 2.1 in [1], $\mathcal{D}(z_0,\delta)$ is the interior of the circle with center at $\frac{1+\delta^2}{1-\delta^2}z_0$ and radius $\frac{2\delta}{|1-\delta^2|}|z_0|$ and the exterior otherwise.

If $\delta\exp(L\sqrt{\frac{\bar{\omega}}{2}}) < 1$, using Lemma 2.1 in [1], we can see that $\mathcal{D}(2\sqrt{i\bar{\omega}}, \delta\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$ is convex. Thus for $\theta \in [0,1]$, we have $\theta\bar{p_1} + (1-\theta)\bar{p_2} \in D^L_\delta$.

If $\delta\exp(L\sqrt{\frac{\bar{\omega}}{2}}) \geq 1$, using Lemma 2.1 in [1], we can see that for $\bar{p}_1$, $\bar{p}_2 \geq 0$, $\theta \in [0,1]$, we have $\theta\bar{p_1} + (1-\theta)\bar{p_2} \in \mathcal{D}(2\sqrt{i\omega}, \delta\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$. Thus for $\theta \in [0,1]$, we have $\theta\bar{p_1} + (1-\theta)\bar{p_2} \in D^L_\delta$.

Therefore D^L_δ is convex.

Suppose that $(\bar{p}^*, h_L(\bar{p}^*))$ is a strictly local minimum of $h_L(\bar{p})$, we prove that it is a global minimum of $h_L(\bar{p})$. Suppose the contrary that there exists $(\bar{p}^{**}, h_L(\bar{p}^{**}))$ such that $h_L(\bar{p}^*) \geq h_L(\bar{p}^{**})$. Then there exists a convex neighborhood U of $\bar{p}^*$, such that $\forall\ s \in U$, $s \neq \bar{p}^*$ and $h_L(s) > h_L(\bar{p}^*)$. Since $\bar{p}^{**} \in D^L_{h_L(\bar{p}^{**})} \subset D^L_{h_L(\bar{p}^*)}$,we have that $\forall\theta \in [0,1]$, $\theta\bar{p}^* + (1-\theta)\bar{p}^{**} \in D^L_{h_L(\bar{p}^*)}$. For θ small enough, we have that $\theta\bar{p}^{**} + (1-\theta)\bar{p}^* \in U$. This is a contradiction.

Thus $\bar{p}^*$ is the unique solution of (2.1.13).

■

Proof of theorem 2.1.1

Case 1: For $L(\bar{\omega}_{max})^{\frac{3}{4}}$ *large and* $L < \sqrt{\frac{10\sqrt{2}-12}{\bar{\omega}_{min}}}$.

Firstly, we will prove that $||\bar{\rho}(\bar{\omega},\bar{p},L)||_\infty = \max\{\bar{\rho}(\bar{\omega}_{min},\bar{p},L), \rho(\bar{\omega}_2,\bar{p},L)\}$ when $\bar{p}$ is closed enough to $\bar{p}_* = (4\bar{\omega}_{min})^{\frac{1}{3}}L^{-\frac{1}{3}}$.

We have that

$$\bar{p}_{\bar{\omega}}(\bar{\omega},\bar{p},L) = -\frac{\sqrt{2}}{2}\exp(-L\sqrt{2\bar{\omega}})\frac{16L\bar{\omega}^2 - 16\bar{p}\bar{\omega} + L\bar{p}^4 + 4\bar{p}^3}{(4\bar{\omega} + 2\sqrt{2\bar{\omega}}\bar{p} + \bar{p}^2)^2}.$$

We consider the function:

$$f(\bar{\omega}) = 16L\bar{\omega}^2 - 16\bar{p}\bar{\omega} + L\bar{p}^4 + 4\bar{p}^3.$$

This is a quadratic equation in $\bar{\omega}$. We will prove that $\Delta' = 64\bar{p}^2 - 16L(L\bar{p}^4 + 4\bar{p}^3) = 16\bar{p}^2(4 - 4L\bar{p} - L^2\bar{p}^2) > 0$. Since $\bar{p}$ is closed to $\bar{p}_*$, we only need to prove that $4 - 4L\bar{p}_* - L^2\bar{p}_*^2 > 0$, or $L\bar{p}_* < 2\sqrt{2} - 2$.

Since $L < \sqrt{\frac{10\sqrt{2}-12}{\bar{\omega}_{min}}}$, we have that

$$L\bar{p}_* = L(4\bar{\omega}_{min})^{\frac{1}{3}}(L)^{-\frac{1}{3}} = (4\bar{\omega}_{min})^{\frac{1}{3}}L^{\frac{2}{3}} < (4\bar{\omega}_{min})^{\frac{1}{3}}(\frac{10\sqrt{2}-14}{\bar{\omega}_{min}})^{\frac{1}{3}} = 2\sqrt{2} - 2.$$

Thus $\Delta' > 0$.

Therefore the equation $f = 0$ has the following solutions:

$$\bar{\omega}_1 = \frac{2\bar{p} - \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L} = \frac{2\bar{p} - \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L},$$

$$\bar{\omega}_2 = \frac{2\bar{p} + \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L} = \frac{2\bar{p} + \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L}.$$

We will prove that in this case $\bar{\omega}_{max} > \bar{\omega}_2(\bar{p})$. In order to do that, we only need to prove that $\bar{\omega}_{max} > \bar{\omega}_2(\bar{p}_*)$.

Since $L(\bar{\omega}_{max})^{\frac{3}{4}}$ is large, then $L > (\frac{\bar{\omega}_{min}}{2\bar{\omega}_{max}^3})^{\frac{1}{4}}$, we have

$$L^4 > \frac{\bar{\omega}_{min}}{2\bar{\omega}_{max}^3}.$$

Thus

$$\bar{\omega}_{max}^3 > \frac{\bar{\omega}_{min}}{2}L^{-4}.$$

Therefore

$$\bar{\omega}_{max} > (\frac{\bar{\omega}_{min}}{2})^{\frac{1}{3}} L^{-\frac{4}{3}} = \frac{(4\bar{\omega}_{min})^{\frac{1}{3}} L^{-\frac{1}{3}}}{2L} = \frac{\bar{p}_*}{2L} = \frac{\bar{\omega}_1 + \bar{\omega}_2}{2}.$$

This means that in order to prove $\bar{\omega}_{max} > \bar{\omega}_2(\bar{p}_*)$, we only have to prove that

$$16L\bar{\omega}_{max}^2 - 16\bar{p}_*\bar{\omega}_{max} + L\bar{p}_*^4 + 4\bar{p}_*^3 > 0.$$

This inequality is equivalent to

$$16L\bar{\omega}_{max}^2 - 16(4\bar{\omega}_{min})^{\frac{1}{3}} L^{-\frac{1}{3}}\bar{\omega}_{max} + L(4\bar{\omega}_{min})^{\frac{4}{3}}(L)^{-\frac{4}{3}} + 16\bar{\omega}_{min}L^{-1} > 0,$$

or

$$\bar{\omega}_{max}^2 L^{\frac{2}{3}})^3 - 2(\frac{\bar{\omega}_{min}}{2})^{\frac{1}{3}}\bar{\omega}_{max}L^{\frac{2}{3}} + (\frac{\bar{\omega}_{min}}{2})^{\frac{4}{3}} L^{\frac{2}{3}} + \bar{\omega}_{min} =$$

$$= \bar{\omega}_{max}L^{\frac{2}{3}}[\bar{\omega}_{max}L^{\frac{4}{3}} - 2(\frac{\bar{\omega}_{min}}{2})^{\frac{1}{3}}] + (\frac{\bar{\omega}_{min}}{2})^{\frac{4}{3}} + \bar{\omega}_{min} > 0.$$

This is true.

Hence $\bar{\omega}_{max} > \bar{\omega}_2(\bar{p}_*)$ and $\bar{\omega}_{max} > \bar{\omega}_2(\bar{p})$ for $\bar{p}$ closed enough to $\bar{p}_*$.

Therefore $||\bar{\rho}(\bar{\omega}, \bar{p}, L)||_\infty = \max\{\bar{\rho}(\bar{\omega}_{min}, \bar{p}, L), \bar{\rho}(\bar{\omega}_2, \bar{p}, L)\}$.

Next, we will prove that $\bar{p}_*$ is an asymptotic solution to the equation $\bar{\rho}(\bar{\omega}_{min}, \bar{p}, L) = \bar{\rho}(\bar{\omega}_2, p, L)$. Let $\bar{p}$ is a number closed to $\bar{p}_*$, and suppose that $\bar{p}$ has the form $\bar{p} \backsim C(\frac{L}{\nu})^{-\gamma}$, we have that

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_{min}, \bar{p}, L) &= \exp(-L\sqrt{2\bar{\omega}_{min}})\frac{(\sqrt{2\bar{\omega}_{min}} - CL^{-\gamma})^2 + 2\bar{\omega}_{min}}{(\sqrt{2\bar{\omega}_{min}} + CL^{-\gamma})^2 + 2\bar{\omega}_{min}} \\
&= \exp(-L\sqrt{2\bar{\omega}_{min}})\frac{4\bar{\omega}_{min} - 2\sqrt{2\bar{\omega}_{min}}CL^{-\gamma} + C^2L^{-2\gamma}}{4\bar{\omega}_{min} + 2\sqrt{2\bar{\omega}_{min}}CL^{-\gamma} + C^2L^{-2\gamma}} \\
&= \exp(-\frac{L}{\nu}\sqrt{2\omega_{min}\nu})\frac{\frac{4\omega_{min}\nu}{C^2}(\frac{L}{\nu})^{2\gamma} - \frac{2\sqrt{2\bar{\omega}_{min}}}{C}L^\gamma + 1}{\frac{4\bar{\omega}_{min}}{C^2}L^{2\gamma} + \frac{2\sqrt{2\bar{\omega}_{min}}}{C}L^\gamma + 1} \\
&\backsim 1 - L\sqrt{2\bar{\omega}_{min}} - \frac{4\sqrt{2\bar{\omega}_{min}}}{C}L^\gamma + \frac{16\bar{\omega}_{min}}{C^2}L^{2\gamma}.
\end{aligned}$$

We also have that

$$\bar{\rho}(\bar{\omega}_2, \bar{p}, L) = \exp(-L\sqrt{2\bar{\omega}_2})\frac{(\sqrt{2\bar{\omega}_2} - CL^{-\gamma})^2 + 2\bar{\omega}_2}{(\sqrt{2\bar{\omega}_2} + CL^{-\gamma})^2 + 2\bar{\omega}_2}.$$

We have

$$\begin{aligned}
\exp(-L\sqrt{2\bar{\omega}_2}) &= \exp(-\sqrt{2L^2\frac{2\bar{p}+\bar{p}\sqrt{4-4L\bar{p}-L^2\bar{p}^2}}{4L}}) \\
&= \exp(-\sqrt{\frac{2\bar{p}L+\bar{p}L\sqrt{4-4L\bar{p}-L^2\bar{p}^2}}{2}}) \\
&\backsim 1-\sqrt{2\bar{p}L} \backsim 1-\sqrt{2C}L^{\frac{1-\gamma}{2}},
\end{aligned}$$

and

$$\frac{(\sqrt{2\bar{\omega}_2}-\bar{p})^2+2\bar{\omega}_2\nu}{(\sqrt{2\bar{\omega}_2}+\bar{p})^2+2\bar{\omega}_2} = \frac{1-\frac{\bar{p}}{\sqrt{2\bar{\omega}_2}}+\frac{\bar{p}^2}{4\bar{\omega}_2}}{1+\frac{\bar{p}}{\sqrt{2\bar{\omega}_2}}+\frac{\bar{p}^2}{4\bar{\omega}_2}} \backsim 1-\frac{2\bar{p}}{\sqrt{2\bar{\omega}_2}}.$$

Moreover, we have that

$$\frac{\bar{p}}{\sqrt{2\bar{\omega}_2}} = \frac{\bar{p}}{\sqrt{\frac{2\bar{p}+\bar{p}\sqrt{4-4L\bar{p}-L^2\bar{p}^2}}{2L}}} = \sqrt{\frac{2\bar{p}L}{2+\sqrt{4-4\bar{p}L-\bar{p}^2L^2}}} \backsim \sqrt{\frac{C}{2}}L^{\frac{1-\gamma}{2}}.$$

Therefore

$$\frac{(\sqrt{2\bar{\omega}_2}-\bar{p})^2+2\bar{\omega}_2}{(\sqrt{2\bar{\omega}_2}+\bar{p})^2+2\bar{\omega}_2} \backsim 1-\sqrt{2C}L^{\frac{1-\gamma}{2}}+CL^{1-\gamma}.$$

Thus

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_2,\bar{p},L) &\backsim (1-\sqrt{2C}L^{\frac{1-\gamma}{2}}+2CL^{1-\gamma})(1-\sqrt{2C}L^{\frac{1-\gamma}{2}}+CL^{1-\gamma} \\
&\backsim 1-2\sqrt{2C}L^{\frac{1-\gamma}{2}}+5CL^{1-\gamma}.
\end{aligned}$$

Equilibrate $\bar{\rho}(\bar{\omega}_2,\bar{p},L)$ and $\bar{\rho}(\bar{\omega}_{min},\bar{p},L)$, we get $\bar{p}_*$.
Finally, we prove that this $\bar{p}_*$ is a stricly local minimum of $||\rho(\bar{\omega},\bar{p},L)||_\infty$.
We have that

$$\frac{\partial}{\partial_{\bar{p}}}\bar{\rho}(\bar{\omega}_2,\bar{p}_*,L) = \frac{4\sqrt{2}\sqrt{\bar{\omega}_2}\exp\left(-L\sqrt{2}\sqrt{\bar{\omega}_2}\right)}{(\bar{p}_*^2+2\sqrt{2}\sqrt{\bar{\omega}_2\nu}\bar{p}_*+4\bar{\omega}_2\nu)^2}(\bar{p}_*^2-4\bar{\omega}_2)\frac{\bar{\omega}_2}{\bar{p}} < 0,$$

and

$$\frac{\partial}{\partial_{\bar{p}}}\bar{\rho}(\bar{\omega}_{min},\bar{p}_*,L) = \frac{4\sqrt{2}\sqrt{\bar{\omega}_{min}}\exp\left(-L\sqrt{2}\sqrt{\bar{\omega}_{min}}\right)}{(\bar{p}_*^2+2\sqrt{2}\sqrt{\bar{\omega}_{min}}\bar{p}_*+4\bar{\omega}_{min})^2}(\bar{p}_*^2-4\bar{\omega}_{min}) > 0.$$

Thus for $\bar{p}$ closed to $\bar{p}_*$, $\bar{p} > \bar{p}_*$, we have that $\max\{\bar{\rho}(\bar{\omega}_2, \bar{p}, L), \bar{\rho}(\bar{\omega}_{min}, \bar{p}, L)\} = \bar{\rho}(\bar{\omega}_{min}, \bar{p}, L) > \bar{\rho}(\bar{\omega}_{min}, \bar{p}_*, L) = \bar{\rho}(\bar{\omega}_2, \bar{p}_*, L)$. And, for $\bar{p}$ closed to $\bar{p}_*$, $\bar{p} < \bar{p}_*$, we have that $\max\{\rho(\bar{\omega}_2, \bar{p}, L), \rho(\bar{\omega}_{min}, \bar{p}, L)\} = \bar{\rho}(\bar{\omega}_2, \bar{p}, L) > \bar{\rho}(\bar{\omega}_{min}, \bar{p}_*, L) = \bar{\rho}(\bar{\omega}_2, \bar{p}_*, L)$. Thus $\bar{p}_*$ is a stricly local minimum of $||\bar{\rho}(\omega, p, L)||_\infty$, then according to Lemma 2.1.1 it is also the global minimum. And

$$\bar{p}_* \backsim (4\bar{\omega}_{min})^{\frac{1}{3}} L^{-\frac{1}{3}},$$

$$||\bar{\rho}(\bar{\omega}, \bar{p}_*, L)||_\infty \backsim 1 - 4(\frac{\bar{\omega}_{min}}{2})^{\frac{1}{6}} L^{\frac{1}{3}}.$$

Case 2: $L\bar{\omega}_{max}^{\frac{3}{4}}$ is small

In this case, we can see that $\bar{\omega}_2 > \bar{\omega}_{max}$. Thus

$$||\bar{\rho}(\bar{\omega}, \bar{p}, L)||_\infty = \max\{\bar{\rho}(\bar{\omega}_{min}, \bar{p}, L), \bar{\rho}(\bar{\omega}_{max}, \bar{p}, L)\}.$$

As in the previous case, we will prove that $\bar{p}_*$ is a solution of the equation $\bar{\rho}(\bar{\omega}_{min}, \bar{p}, L) = \bar{\rho}(\bar{\omega}_{max}, \bar{p}, L)$.

Let $\bar{p}$ be a number closed enough to $\bar{p}_*$. We have that

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_{min}, \bar{p}, L) &= \exp(-L\sqrt{2\bar{\omega}_{min}})\frac{(\sqrt{2\bar{\omega}_{min}} - \bar{p})^2 + 2\bar{\omega}_{min}}{(\sqrt{2\bar{\omega}_{min}} + \bar{p})^2 + 2\bar{\omega}_{min}\nu} \\
&\backsim (1 - L\sqrt{2\bar{\omega}_{min}} + L^2 2\bar{\omega}_{min})(1 - 4\frac{\sqrt{2\bar{\omega}_{min}}}{\bar{p}} + \frac{16\bar{\omega}_{min}}{\bar{p}^2}).
\end{aligned}$$

Since $L\bar{\omega}_{max}^{\frac{3}{4}}$ is small, then $L < (\frac{\bar{\omega}_{min}}{2\bar{\omega}_{max}^3})^{\frac{1}{4}}$, we have that

$$L\sqrt{2\bar{\omega}_{min}} < (\frac{\bar{\omega}_{min}}{2\bar{\omega}_{max}^3})^{\frac{1}{4}}\sqrt{2\bar{\omega}_{min}} = 2^{\frac{1}{4}}(\frac{\bar{\omega}_{min}}{\bar{\omega}_{max}})^{\frac{3}{4}},$$

and

$$\sqrt{\frac{\bar{\omega}_{min}}{\bar{p}^2}} \backsim \sqrt{\frac{\bar{\omega}_{min}}{4\sqrt{\bar{\omega}_{min}\bar{\omega}_{max}}}} = \frac{1}{2}(\frac{\bar{\omega}_{min}}{\bar{\omega}_{max}})^{\frac{1}{4}}.$$

Therefore

$$\bar{\rho}(\bar{\omega}_{min}, \bar{p}, L) \backsim 1 - \frac{4\sqrt{2\bar{\omega}_{min}}}{\bar{p}}.$$

We can suppose that $4\bar{\omega}_{min} < \bar{\omega}_{max}$, then $2(\bar{\omega}_{min}\bar{\omega}_{max})^{\frac{1}{4}} < \sqrt{2\bar{\omega}_{max}}$. Thus $\bar{p} < \sqrt{2\bar{\omega}_{max}}$. Hence

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_{max}, \bar{p}, L) &= \exp(-L\sqrt{2\bar{\omega}_{max}})\frac{(\sqrt{2\bar{\omega}_{max}} - \bar{p})^2 + 2\bar{\omega}_{max}}{(\sqrt{2\bar{\omega}_{max}} + \bar{p})^2 + 2\bar{\omega}_{max}} \\
&= \exp(-L\sqrt{2\bar{\omega}_{max}})\frac{(1 - \frac{\bar{p}}{\sqrt{2\bar{\omega}_{max}}})^2 + 1}{(1 + \frac{\bar{p}}{\sqrt{2\bar{\omega}_{max}}})^2 + 1} \\
&= \exp(-L\sqrt{2\bar{\omega}_{max}})\frac{1 - \frac{\bar{p}}{\sqrt{2\bar{\omega}_{max}}} + \frac{\bar{p}^2}{4\bar{\omega}_{max}}}{1 + \frac{\bar{p}}{\sqrt{2\bar{\omega}_{max}}} + \frac{\bar{p}^2}{4\bar{\omega}_{max}}} \\
&\backsim (1 - L\sqrt{2\bar{\omega}_{max}} + L^2 2\bar{\omega}_{max})(1 - \frac{2\bar{p}}{\sqrt{2\bar{\omega}_{max}}} + \frac{\bar{p}^2}{\bar{\omega}_{max}}).
\end{aligned}$$

Since $L \backsim C\bar{\omega}_{max}^{-\frac{3}{4}+\gamma}$, then

$$L\sqrt{2\bar{\omega}_{max}} < C(\frac{\bar{\omega}_{min}}{2\bar{\omega}_{max}^3})^{\frac{1}{4}}\sqrt{2\bar{\omega}_{max}}\bar{\omega}_{max}^{\gamma} = C(\bar{\omega}_{min})^{\frac{1}{4}}2^{\frac{1}{4}}\bar{\omega}_{max}^{-\frac{1}{4}+\gamma}.$$

We have also

$$\frac{\bar{p}^2}{\bar{\omega}_{max}} \backsim \frac{4\sqrt{\bar{\omega}_{min}\bar{\omega}_{max}}}{\bar{\omega}_{max}} = 4\sqrt{\frac{\bar{\omega}_{min}}{\bar{\omega}_{max}}}.$$

Therefore

$$\bar{\rho}(\bar{\omega}_{max}, \bar{p}, L) \backsim 1 - \frac{2\bar{p}}{\sqrt{2\bar{\omega}_{max}}}.$$

Equilibrate the two asymptotic expansion $\bar{\rho}(\bar{\omega}_{max}, \bar{p}, L)$ and $\bar{\rho}(\bar{\omega}_2, \bar{p}, L)$ we have the equation

$$\frac{2\bar{p}}{\sqrt{2\bar{\omega}_{max}}} \backsim \frac{4\sqrt{2\bar{\omega}_{min}}}{\bar{p}}.$$

Thus $\bar{p} \backsim 2(\bar{\omega}_{min}\bar{\omega}_{max})^{\frac{1}{4}}$ or $\bar{p}_*$ is an asymptotic solution of the equation $\bar{\rho}(\bar{\omega}_{max}, \bar{p}, L) = \bar{\rho}(\bar{\omega}_2, \bar{p}, L)$. Using the same argument as in the previous section we have that this $\bar{p}_*$ is a global minimum of $||\bar{\rho}(\bar{\omega}, \bar{p}, L)||_\infty$. And

$$||\bar{\rho}(\bar{\omega}_{min}, \bar{p}_*, L)||_\infty \backsim 1 - \frac{2\bar{p}_*}{\sqrt{2\bar{\omega}_{max}}} \backsim 1 - \frac{4(\bar{\omega}_{min}\bar{\omega}_{max})^{\frac{1}{4}}}{\sqrt{2\bar{\omega}_{max}}} \backsim 1 - 2\sqrt{2}(\frac{\bar{\omega}_{min}}{\bar{\omega}_{max}})^{\frac{1}{4}}.$$

■

Proof of Theorem 2.1.2

Case 1: $L = C_1\Delta x$, $\Delta t = C_2\Delta x$, $\Delta x < \min\left\{\left(\frac{(17-12\sqrt{2})2TC_2}{\pi^2(C_1\nu^{-\frac{1}{2}})^4}\right)^{\frac{1}{3}}, \frac{C_2^3}{\pi^2(C_1\nu^{-\frac{1}{2}})^4 2T}\right\}$.

Firstly, we prove that $||\bar{\rho}(\bar{\omega};\bar{p},L)||_\infty = \max\{\bar{\rho}(\frac{\pi}{2T\nu};\bar{p},L), \rho(\frac{\pi}{\Delta t\nu};\bar{p},L)\}$ when $\bar{p}$ is closed to $\bar{p}_*$.

We have that

$$\partial_{\bar{\omega}}\bar{p}(\bar{\omega};\bar{p},L) = -\frac{\sqrt{2}}{2}\exp(-L\sqrt{2\bar{\omega}})\frac{16L\bar{\omega}^2 - 16\bar{p}\bar{\omega} + L\bar{p}^4 + 4\bar{p}^3}{(4\bar{\omega} + 2\sqrt{2\bar{\omega}}\bar{p} + \bar{p}^2)^2}.$$

The function $16L\bar{\omega}^2 - 16\bar{p}\bar{\omega} + L\bar{p}^4 + 4\bar{p}^3$, is a quadratic function of $\bar{\omega}$ and it has $\Delta'(\bar{p}) = 16\bar{p}^2(4 - 4L\bar{p} - L^2\bar{p}^2)$. We will prove that $\Delta'(\bar{p}) > 0$. Since $\bar{p}$ is closed to $\bar{p}_*$, we only need to prove that $\Delta'(\bar{p}_*) > 0$. We have that

$$\Delta x < \left(\frac{(17-12\sqrt{2})2TC_2}{\pi^2(C_1\nu^{-\frac{1}{2}})^4}\right)^{\frac{1}{3}}.$$

This implies

$$\Delta x < \frac{(\sqrt{2}-1)^{\frac{4}{3}}(2TC_2)^{\frac{1}{3}}}{(\sqrt{\pi}C_1\nu^{-\frac{1}{2}})^{\frac{4}{3}}}.$$

Therefore

$$\frac{2\sqrt{\pi\nu}C_1\nu^{-1}}{(2TC_2)^{\frac{1}{4}}}\Delta x^{\frac{3}{4}} < 2\sqrt{2} - 2,$$

or

$$L\bar{p}_* < 2\sqrt{2} - 2.$$

Hence $\Delta'(\bar{p}_*) > 0$.

Therefore the equation $f = 0$ has the following solutions:

$$\bar{\omega}_1 = \frac{2\bar{p} - \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L} = \frac{2\bar{p} - \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L},$$

$$\bar{\omega}_2 = \frac{2\bar{p} + \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L} = \frac{2\bar{p} + \bar{p}\sqrt{4 - 4L\bar{p} - L^2\bar{p}^2}}{4L}.$$

We prove that for $\bar{p}$ closed to $\bar{p}_*$, we also have $\frac{\bar{p}}{2L} > \frac{\pi}{\Delta t\nu}$, which implies $\bar{\omega}_2 > \frac{\pi}{\Delta t\nu}$. In fact, we only need to prove that $\frac{\bar{p}_*}{2L} > \frac{\pi}{\Delta t\nu}$.

Since

$$\Delta x < \frac{C_2^3}{\pi^2(C_1\nu^{-\frac{1}{2}})^4 2T},$$

we have

$$\frac{\frac{2\sqrt{\pi\nu}}{(2TC_2)^{\frac{1}{4}}}\Delta x^{-\frac{1}{4}}}{2C_1\Delta x} > \frac{\pi}{C_2\Delta x},$$

or $\frac{\bar{p}}{2L} > \frac{\pi}{\Delta t\nu}$.

Therefore

$$||\bar{\rho}(\bar{\omega},\bar{p},L)||_\infty = \max\{\bar{\rho}(\frac{\pi}{2T\nu};\bar{p},L),\bar{\rho}(\frac{\pi}{\Delta t\nu};\bar{p},L)\}.$$

Using the same argument as in Theorem 2.1.1, we have that problem (2.1.13) has a unique solution

$$\bar{p}_* \backsim \frac{2\sqrt{\pi}}{(2TC_2)^{\frac{1}{4}}}\Delta x^{-\frac{1}{4}},$$

then

$$||\bar{\rho}(\bar{\omega},\bar{p}_*,L)||_\infty = 1 - \frac{2\sqrt{2}C_2^{\frac{1}{4}}}{(2T)^{\frac{1}{4}}}\Delta x^{\frac{1}{4}} + O(\Delta x^{\frac{1}{2}}).$$

Case 2: $L = C_1\Delta x$, $\Delta t = C_2\Delta x^2$, $\Delta x \leq \min\big\{\big(\frac{\pi^2 2T(C_1\nu^{-1})^4}{4C_2^3}\big)^{\frac{1}{2}}, \big(\frac{(10\sqrt{2}-14)2T}{\pi(C_1\nu^{-1})^2}\big)^{\frac{1}{2}}\big\}$.

Firstly, we prove that $||\bar{\rho}(\frac{\pi}{2T\nu},\bar{p},L)||_\infty = \max\{\bar{\rho}(\frac{\pi}{2T\nu},\bar{p},L),\bar{\rho}(\bar{\omega}_2,\bar{p},L)\}$ when $\bar{p}$ is closed to $\bar{p}_*$.

We have that

$$\partial_{\bar{\omega}}\bar{p}(\bar{\omega},\bar{p},L) = -\frac{\sqrt{2}}{2}\exp(-L\sqrt{2\bar{\omega}})\frac{16L\bar{\omega}^2 - 16\bar{p}\bar{\omega} + L\bar{p}^4 + 4\bar{p}^3}{(4\bar{\omega} + 2\sqrt{2\bar{\omega}}\bar{p} + \bar{p}^2)^2}.$$

Similar as in the previous case, we prove that $\Delta'(\bar{p}_*) = 16\bar{p}_*^2(4 - 4L\bar{p}_* - L^2\bar{p}_*^2) > 0$.

We have that

$$\Delta x \leq \big(\frac{(10\sqrt{2}-14)2T}{\pi(C_1\nu^{-1})^2}\big)^{\frac{1}{2}}.$$

This leads to

$$\frac{4\pi(C_1\nu^{-1})^2}{2T}\Delta x^2 < 8(5\sqrt{2}-7).$$

Thus

$$\frac{4\pi(C_1\nu^{-1})^2}{2T}\Delta x^2 < 8(5\sqrt{2}-7) = 8(\sqrt{2}-1)^3.$$

Hence

$$(\frac{4\pi(C_1\nu)^2}{2T})^{\frac{1}{3}}\Delta x^{\frac{2}{3}} < 2\sqrt{2}-2,$$

or $L\bar{p}_* < 2\sqrt{2}-2$, and $\Delta' > 0$.

We prove that in this case, $\bar{\omega}_2 < \frac{\pi}{\Delta t\nu}$.

We have

$$\Delta x \le (\frac{\pi^2 2T(C_1\nu^{-1})^4}{4C_2^3})^{\frac{1}{2}}.$$

Hence

$$\frac{2\pi}{T(C_1\nu^{-1})^4}\Delta x^2 \le \frac{\pi^3}{C_2^3}.$$

Thus

$$\frac{(\frac{4\pi\nu^2}{2TC_1})^{\frac{1}{3}}\Delta x^{-\frac{1}{3}}}{C_1\Delta x} \le \frac{\pi}{C_2}\Delta x^{-2}.$$

Therefore $\frac{\bar{p}}{L} \le \frac{\pi}{\Delta t\nu}$, which means $\bar{\omega}_2 < \frac{\pi}{\Delta t\nu}$.

Therefore

$$||\bar{\rho}(\bar{\omega},\bar{p},L)||_\infty = \max\{\bar{\rho}(\frac{\pi}{2T\nu};\bar{p},L),\bar{\rho}(\bar{\omega}_2;\bar{p},L)\}.$$

Equilibrate $\bar{\rho}(\frac{\pi}{2T\nu},\bar{p},L)$ and $\bar{\rho}(\bar{\omega}_2,\bar{p},L)$, we get $\bar{p}_*$ and using the same argument of the previous case, we can conclude that problem (2.1.13) has a unique solution

$$\bar{p}_* \backsim (\frac{2\pi}{TC_1})^{\frac{1}{3}}\Delta x^{-\frac{1}{3}},$$

then

$$||\bar{\rho}(\bar{\omega},\bar{p}_*,L)||_\infty = 1-4(\frac{\pi C_1^2}{4T})^{\frac{1}{6}}\Delta x^{\frac{1}{3}} + O(\Delta x^{\frac{2}{3}}).$$

■

2.1.2 Proof of the Theorems in the Non-Overlapping Case

Proof of Theorem 2.1.3

Since $\omega \in [\frac{\pi}{2T},\frac{\pi}{\Delta t}]$, then $x = \frac{\sqrt{2\omega\nu}}{p}$ belongs to $[\sqrt{\frac{2\pi\nu}{p^2 2T}},\sqrt{\frac{2\pi\nu}{p^2\Delta t}}]$.

Thus

$$\rho(\omega) = f(x) = \frac{2x^2-2x+1}{2x^2+2x+1}. \tag{2.1.14}$$

We have

$$f'(x) = \frac{4(2x^2-1)}{(2x^2+2x+1)^2}. \tag{2.1.15}$$

We have the following cases

Case 1: $\sqrt{\frac{2\nu\pi}{p^2 2T}} \geq \frac{1}{\sqrt{2}}$.

We have that

$$\max_{\omega} \rho = f(\sqrt{\frac{2\pi\nu}{p^2\Delta t}}), \tag{2.1.16}$$

Since $\sqrt{\frac{2\nu\pi}{p^2 2T}} \geq \frac{1}{\sqrt{2}}$, we have that $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} \geq \frac{1}{\sqrt{2}}\sqrt{\frac{2T}{\Delta t}} > \frac{1}{\sqrt{2}}$.

Thus

$$\min_p \max_{\omega} \rho = f(\frac{1}{\sqrt{2}}\sqrt{\frac{2T}{\Delta t}}), \tag{2.1.17}$$

when $\omega = \frac{\pi}{\Delta t}$ and $p = 2\sqrt{\frac{\nu\pi}{2T}}$.

Case 2: $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} \leq \frac{1}{\sqrt{2}}$.

We have that

$$\max_{\omega} \rho = f(\sqrt{\frac{\pi\nu}{p^2 T}}), \tag{2.1.18}$$

Since $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} \leq \frac{1}{\sqrt{2}}$, we have that $\sqrt{\frac{\nu\pi}{p^2 T}} \leq \frac{1}{\sqrt{2}}\sqrt{\frac{\Delta t}{2T}} < \frac{1}{\sqrt{2}}$.

Thus

$$\min_p \max_{\omega} \rho = f(\frac{1}{\sqrt{2}}\sqrt{\frac{\Delta t}{2T}}) = f(\frac{1}{\sqrt{2}}\sqrt{\frac{2T}{\Delta t}}), \tag{2.1.19}$$

when $\omega = \frac{\pi}{2T}$ and $p = 2\sqrt{\frac{\nu\pi}{\Delta t}}$.

Case 3: $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} > \frac{1}{\sqrt{2}} > \sqrt{\frac{2\nu\pi}{p^2 2T}}$.

We can see that if $x > y$ and $2xy > 1$, $f(x) > f(y)$; and if $x > y$ and $2xy < 1$, $f(x) < f(y)$; and if $x > y$ and $2xy = 1$, $f(x) = f(y)$.

$$\max_{\omega} \rho = \max\{f(\sqrt{\frac{2\pi\nu}{p^2 2T}}), f(\sqrt{\frac{2\pi\nu}{p^2\Delta t}})\}. \tag{2.1.20}$$

Case 3.1: $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} \cdot \sqrt{\frac{2\nu\pi}{p^2 2T}} \geq \frac{1}{2}$, or $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} \cdot \sqrt{\frac{2\nu\pi}{p^2\Delta t}} \geq \frac{1}{2}\sqrt{\frac{2T}{\Delta t}}$, or $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} \geq \frac{1}{\sqrt{2}}(\frac{2T}{\Delta t})^{\frac{1}{4}} > \frac{1}{\sqrt{2}}$.

$$\max_{\omega} \rho = f(\sqrt{\frac{2\pi\nu}{p^2\Delta t}}). \tag{2.1.21}$$

Thus

$$\min_p \max_\omega \rho = f(\frac{1}{\sqrt{2}}(\frac{2T}{\Delta t})^{\frac{1}{4}}) < f(\frac{1}{\sqrt{2}}\sqrt{\frac{2T}{\Delta t}}), \tag{2.1.22}$$

when $\omega = \frac{\pi}{\Delta t}$ and $p = 2\frac{\sqrt{\nu\pi}}{(\Delta t 2T)^{\frac{1}{4}}}$.

Case 3.2: $\sqrt{\frac{2\nu\pi}{p^2\Delta t}} \cdot \sqrt{\frac{\nu\pi}{p^2 T}} \leq \frac{1}{2}$, or $\sqrt{\frac{\nu\pi}{p^2 T}} \cdot \sqrt{\frac{\nu\pi}{p^2 T}} \leq \frac{1}{2}\sqrt{\frac{\Delta t}{2T}}$, or $\sqrt{\frac{\nu\pi}{p^2 T}} \leq \frac{1}{\sqrt{2}}(\frac{\Delta t}{2T})^{\frac{1}{4}} < \frac{1}{\sqrt{2}}$.

$$\max_\omega \rho = f(\sqrt{\frac{\pi\nu}{p^2 T}}). \tag{2.1.23}$$

Thus

$$\min_p \max_\omega \rho = f(\frac{1}{\sqrt{2}}(\frac{\Delta t}{2T})^{\frac{1}{4}}) = f(\frac{1}{\sqrt{2}}(\frac{2T}{\Delta t})^{\frac{1}{4}}) < f(\frac{1}{\sqrt{2}}\sqrt{\frac{2T}{\Delta t}}), \tag{2.1.24}$$

when $\omega = \frac{\pi}{2T}$ and $p = 2\frac{\sqrt{\nu\pi}}{(\Delta t 2T)^{\frac{1}{4}}}$.

$\blacksquare$

2.2 Optimized Schwarz Waveform Relaxation Methods For One Dimensional Heat Equation With First Order Transmission Condition

In this chapter, we consider the following algorithm

$$\begin{cases} (\partial_t - \nu\partial_{xx})u_1^k = f & \text{in } \Omega_1 \times (0,T), \\ u_1^k(x,0) = u_0(x) & \text{in } \Omega_1, \\ (\partial_x + \frac{p}{2\nu} + 2q\partial_t)u_1^k(L,.) = (\partial_x + \frac{p}{2\nu} + 2q\partial_t)u_2^{k-1}(L,.) & \text{in } (0,T), \end{cases} \tag{2.2.1}$$

$$\begin{cases} (\partial_t - \nu\partial_{xx})u_2^k = f & \text{in } \Omega_2 \times (0,T), \\ u_2^k(x,0) = u_0(x) & \text{in } \Omega_2, \\ (\partial_x - \frac{p}{2\nu} - 2q\partial_t)u_2^k(0,.) = (\partial_x - \frac{p}{2\nu} - 2q\partial_t)u_1^{k-1}(0,.) & \text{in } (0,T). \end{cases}$$

Similar as in the previous chapter, we consider the following problem

$$\begin{cases} \partial_t e_1 - \nu\partial_{xx}e_1 = 0 & \text{in } \Omega_1 \times (0,T), \\ e_1(x,0) = u_0(x) & \text{in } \Omega_1, \\ (\partial_x + \frac{p}{2\nu} + 2q\partial_t)e_1^k(L,.) = (\partial_x + \frac{p}{2\nu} + 2q\partial_t)e_2(L,.) & \text{in } (0,T), \end{cases} \tag{2.2.2}$$

$$\begin{cases} \partial_t e_2 - \nu\partial_{xx}e_2 = 0 & \text{in } \Omega_2 \times (0,T), \\ e_2(x,0) = u_0(x) & \text{in } \Omega_2, \\ (\partial_x - \frac{p}{2\nu} - 2q\partial_t)e_2(0,.) = (\partial_x - \frac{p}{2\nu} - 2q\partial_t)e_1(0,.) & \text{in } (0,T). \end{cases}$$

From (2.2.2), we have that

$$i\omega\mathfrak{F}e_1 - \nu\partial_{xx}\mathfrak{F}e_1 = 0.$$

Therefore

$$\mathfrak{F}e_1 = C_1\exp(\sqrt{\frac{i\omega}{\nu}}x) + C_2\exp(-\sqrt{\frac{i\omega}{\nu}}x),$$

where $Re(\sqrt{\frac{i\omega}{\nu}}) \geq 0$.
Since $x \in (-\infty, L)$ and $\mathfrak{F}e_1(x, .) \in L^2(\mathbb{R})$, we have $C_2 = 0$. Thus

$$\mathfrak{F}e_1 = C_1 \exp(\sqrt{\frac{i\omega}{\nu}}x).$$

From (2.2.2), we have that

$$\partial_x \mathfrak{F}e_1(L, \omega) + \frac{p}{2\nu}\mathfrak{e}_1(L, \omega) + 2qi\omega\mathfrak{F}e_1(L, \omega) = \mathfrak{F}h_L(\omega).$$

Thus

$$(C_1\sqrt{\frac{i\omega}{\nu}} + C_1\frac{p}{2\nu} + C_1 2q\omega i)\exp(\sqrt{\frac{i\omega}{\nu}}L) = \mathfrak{F}h_L(\omega).$$

Hence

$$C_1\frac{\sqrt{4\nu\omega i} + p + 4q\omega\nu i}{2\nu} = \mathfrak{F}h_L \exp(-\sqrt{\frac{i\omega}{\nu}}L).$$

Thus

$$C_1 = \frac{2\nu}{\sqrt{4\nu\omega i} + p + 4q\omega\nu i}\mathfrak{F}h_L \exp(-\sqrt{\frac{i\omega}{\nu}}L).$$

Therefore

$$\mathfrak{F}e_1 = \frac{2\nu}{\sqrt{4\nu\omega i} + p + 4q\omega\nu i}\mathfrak{F}h_L \exp\exp(\sqrt{\frac{i\omega}{\nu}}(x - L)).$$

From (2.2.2), we have that

$$i\omega\mathfrak{F}e_2 - \nu\partial_{xx}\mathfrak{F}e_2 = 0.$$

Therefore

$$\mathfrak{F}e_2 = D_1 \exp(\sqrt{\frac{i\omega}{\nu}}x) + D_2 \exp(-\sqrt{\frac{i\omega}{\nu}}x),$$

where $Re(\sqrt{\frac{i\omega}{\nu}}) \geq 0$.
Since $x \in (0, \infty)$ and $\mathfrak{F}e_2(x, .) \in L^2(\mathbb{R})$, we have $D_1 = 0$. Thus

$$\mathfrak{F}e_2 = D_2 \exp(-\sqrt{\frac{i\omega}{\nu}}x).$$

From (2.2.2), we have that

$$\partial_x \mathfrak{F}e_2(0,\omega) - \frac{p}{2\nu}\mathfrak{F}e_2(0,\omega) - \frac{2q}{\nu}i\omega\mathfrak{F}e_2(0,\omega) = \mathfrak{h}_\mathfrak{o}(\omega).$$

Thus

$$(D_2\sqrt{\frac{i\omega}{\nu}} - D_2\frac{p}{2\nu} - D_2 2q\omega i) = \mathfrak{F}h_0(\omega).$$

Hence

$$D_2\frac{\sqrt{4\nu\omega i} - p - 4q\omega i}{2} = \mathfrak{F}h_0.$$

Thus

$$D_2 = \frac{2\nu}{\sqrt{4\nu\omega i} - p - 4q\omega\nu i}\mathfrak{F}h_0.$$

Therefore

$$\mathfrak{F}e_2 = \frac{2\nu}{\sqrt{4\nu\omega i} - p - 4q\omega\nu i}\mathfrak{F}h_0 \exp(\sqrt{-\frac{i\omega}{\nu}}x).$$

Similar as in the previous chapter, we can define the convergence factor as

$$\rho(\omega,p,q,L) = |\frac{2\sqrt{i\omega\nu} - p - 4q\omega\nu i}{2\sqrt{i\omega\nu} + p + 4q\omega\nu i}\exp(-\sqrt{i\omega\nu}\frac{L}{\nu})|^2.$$

Put $\bar{\omega} = \frac{\omega}{\nu}$, $\bar{p} = \frac{p}{\nu}$ and $\bar{\rho}(\bar{\omega},\bar{p},q,L) = \rho(\omega,p,q,L)$, we need to solve the problem

$$\min_{\bar{p},q\in\mathbb{R}} \max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega},\bar{p},q,L). \tag{2.2.3}$$

Similar as in the previous chapter, we only need to solve the following problem

$$\min_{\bar{p},q\geq 0} \max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega},\bar{p},q,L). \tag{2.2.4}$$

We have the following theorems for the Overlapping Case

Theorem 2.2.1. *If we fix $\bar{\omega}_{min}$ and $\bar{\omega}_{max}$, then for L small satisfying that $L^{\frac{8}{5}}\bar{\omega}_{max}$ is small and $L\bar{\omega}_{max}$ is not small, the problem (2.2.4) has a unique solution*

$$\bar{p}_* \backsim \sqrt{2}\bar{\omega}_{min}^{\frac{3}{8}}\bar{\omega}_{max}^{\frac{1}{8}},$$

$$q_* \backsim \sqrt{2}\bar{\omega}_{min}^{-\frac{1}{8}}\bar{\omega}_{max}^{-\frac{3}{8}},$$

$$\min_{\bar{p},q\geq 0}\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]}\bar{\rho}(\bar{\omega},\bar{p},q,L) \backsim 1-4\bar{\omega}_{min}^{\frac{1}{8}}\bar{\omega}_{max}^{-\frac{1}{8}}.$$

Theorem 2.2.2.

Case 1: For Δx small enough $L = C_1\Delta x$, $\Delta t = C_2\Delta x$, $\bar{\omega} \in [\frac{\pi}{2T\nu}, \frac{\pi}{C_2\nu}\Delta x^{-1}]$.

There exists a unique pair $(\bar{p}_*, q_*)$ such that $\min_{\bar{p},q\geq 0}\max_{\bar{\omega}\in[\frac{\pi}{2T\nu},\frac{\pi}{\Delta t\nu}]}\bar{\rho}(\bar{\omega},\bar{p},q,L) = \max_{\bar{\omega}\in[\frac{\pi}{2T\nu},\frac{\pi}{\Delta t\nu}]}\bar{\rho}(\bar{\omega},\bar{p}_*,q_*,L)$. Then

$$\bar{p}_* \backsim \sqrt{2}(\frac{\pi^4}{T^3C_2\nu^4})^{\frac{1}{8}}\Delta x^{-\frac{1}{8}},$$

$$q_* \backsim \sqrt{2}(\frac{\pi^4\nu^4}{TC_2^3})^{-\frac{1}{8}}\Delta x^{\frac{3}{8}},$$

$$\min_{\bar{p},q\geq 0}\max_{\bar{\omega}\in[\frac{\pi}{2T\nu},\frac{\pi}{\Delta t\nu}]}\bar{\rho}(\bar{\omega},\bar{p},q,L) \backsim 1-4(\frac{C_2}{2T})^{\frac{1}{8}}\Delta x^{\frac{1}{8}}+O(\Delta x^{\frac{1}{4}}).$$

Case 2: For Δx small enough $L = C_1\Delta x$, $\Delta t = C_2\Delta x^2$, $\bar{\omega} \in [\frac{\pi}{2T\nu}, \frac{\pi}{C_2\nu^4}\Delta x^{-2}]$.

There exists a unique pair $(\bar{p}_*, q_*)$ such that $\min_{\bar{p},q\geq 0}\max_{\omega\in[\frac{\pi}{2T\nu},\frac{\pi}{\Delta t\nu}]}\bar{\rho}(\bar{\omega},\bar{p},q,L) = \max_{\bar{\omega}\in[\frac{\pi}{2T\nu},\frac{\pi}{\Delta t\nu}]}\bar{\rho}(\bar{\omega},\bar{p}_*,q_*,L)$. Then

$$\bar{p}_* \backsim (4\frac{\pi^2\nu^2}{4T^2}(C_1\nu^{-1})^{-1})^{\frac{1}{5}}\Delta x^{-\frac{1}{5}},$$

$$q_* \backsim (16(C_1\nu^{-1})^3\frac{2T}{\pi\nu})^{\frac{1}{5}}\Delta x^{\frac{3}{5}},$$

$$\min_{\bar{p},q\geq 0}\max_{\bar{\omega}\in[\frac{\pi}{2T\nu},\frac{\pi}{\Delta t\nu}]}\bar{\rho}(\bar{\omega},\bar{p},q,L) \backsim 1-2^{\frac{21}{10}}(\frac{\pi\nu}{2T})^{\frac{1}{10}}(C_1\nu^{-1})^{\frac{1}{5}}\Delta x^{\frac{1}{5}}+O(\Delta x^{\frac{2}{5}}).$$

Remark 2.2.1.

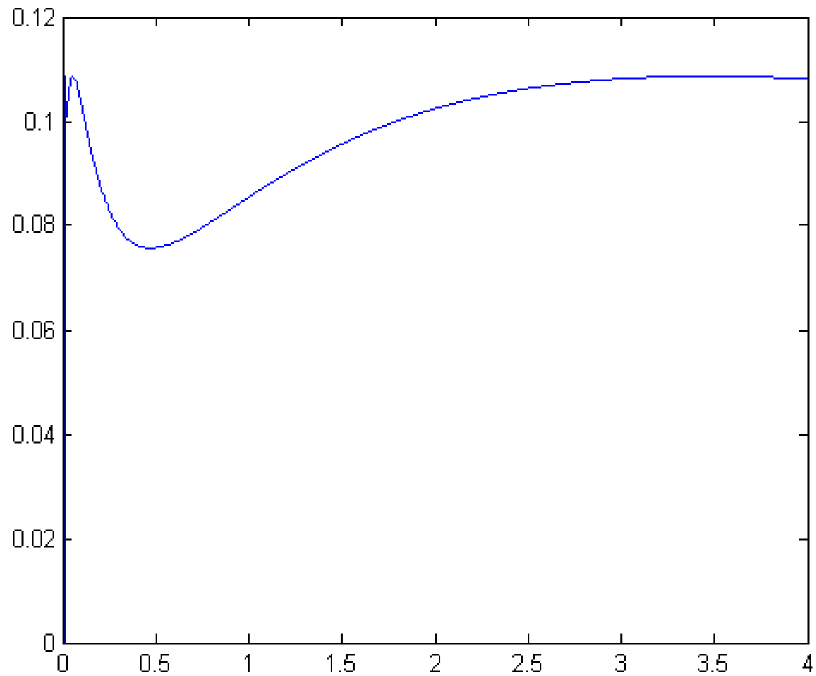


Figure 2.2.1.

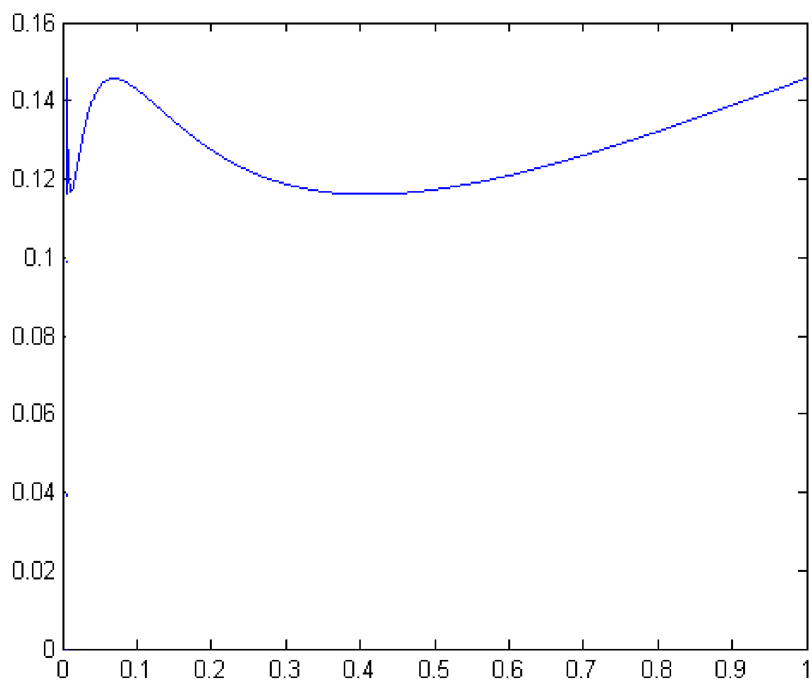


Figure 2.2.2.

Figure 3.1 is the graph of $\bar{\rho}$ with respect to $\bar{\omega}$ for some $\bar{p}$. In the second case of the previous two theorems, we can prove that the solution $(\bar{p}_*, q_*)$ of (2.2.3) *can be obtained by equilibrating the there points on the graph: the boundary point on the left and the two maximal points (with respect to $\bar{\omega}_{min}$ and the maximum point $\bar{\omega}_2$, $\bar{\omega}_4$ of $\bar{\rho}$) on the graph. In the first case $\bar{\omega}_4 > \bar{\omega}_{max}$, we equilibrate the two boundaries and the maximal point $\bar{\omega}_2$ to get $(\bar{p}_*, q_*)$.*

We have the following theorem for the nonoverlapping case

Theorem 2.2.3. *The equation* (2.2.4) *has a unique solution*

$$\bar{p}_* = \sqrt{2}\bar{\omega}_{min}^{\frac{3}{8}}\bar{\omega}_{max}^{\frac{1}{8}} = \sqrt{2}(\frac{\pi\nu^{-1}}{2T})^{\frac{3}{8}}(\pi\nu^{-1})^{\frac{1}{8}}\Delta t^{-\frac{1}{8}},$$

and

$$q_* = \sqrt{2}(\omega_{min}\nu)^{-\frac{1}{8}}(\omega_{max}\nu)^{-\frac{3}{8}} = \sqrt{2}(\frac{\pi\nu}{2T})^{-\frac{1}{8}}(\pi\nu)^{-\frac{3}{8}}\Delta t^{\frac{3}{8}}.$$

Then, we have that

$$\min_{\bar{p},q\geq 0}\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]}\rho(\bar{\omega},\bar{p},q) = \max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]}\rho(\bar{\omega},\bar{p}_*,q_*) \backsim 1 - 4\bar{\omega}_{min}^{\frac{1}{8}}\bar{\omega}_{max}^{-\frac{1}{8}}$$
$$\backsim 1 - 4(\frac{1}{2T})^{\frac{1}{8}}\Delta t^{\frac{1}{8}}.$$

2.2.1 Proof of the Theorems in the Overlapping Case

In this section, we will consider the problem of optimizing $(\bar{p}, q)$ in the overlapping case.

Putting

$$h_L(\bar{p}, q) = \max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]}\bar{\rho}(\bar{\omega},\bar{p},q,L) = ||\bar{\rho}(\bar{\omega},\bar{p},q,L)||_\infty,$$

we call that $(\bar{p}^*, q^*, h_L(\bar{p}^*, q^*))$ is a strictly local minimum of $h_L(\bar{p}, q)$ if and only if there exists ϵ_1, ϵ_2 positive such that for all $(\bar{p}, q)$ in $(\bar{p}^* - \epsilon_1, \bar{p}^* + \epsilon_1) \times (q^* - \epsilon_2, q^* + \epsilon_2)$, we have $h_L(\bar{p}, q) < h_L(\bar{p}^*, q^*)$.

In order to prove the theorems, we need the following lemma:

Lemma 2.2.1. *If $(\bar{p}^*, q^*, h_L(\bar{p}^*, q^*))$ is a strictly local minimum of $h_L(\bar{p}, q)$, then it is the global minimum of $h_L(\bar{p}, q)$ and $(\bar{p}^*, q^*)$ is the unique solution of* (2.2.4).

Proof of Lemma 2.2.1

We denote $\mathcal{D}(z_0,\delta) = \{z \in \mathbb{C}, |\frac{z-z_0}{z+z_0}| < \delta\}$, and $D^L_\delta = \{(\bar{p},q)|h_L(\bar{p},q) \leq \delta\}$.

We first prove that D^L_δ is a convex set. Let $(\bar{p}_1,q_1)$ and $(\bar{p}_2,q_2)$ be to elements of D^L_δ, we have that

$$||\frac{2\sqrt{i\bar{\omega}} - \bar{p} - q\bar{\omega}i}{2\sqrt{i\bar{\omega}} + \bar{p} + q\bar{\omega}i}\exp(-\sqrt{i\bar{\omega}}\frac{L}{\nu})||_\infty \leq \sqrt{\delta}.$$

Thus $\forall \bar{\omega} \in [\bar{\omega}_{min}, \bar{\omega}_{max}]$,

$$|\frac{2\sqrt{i\bar{\omega}} - \bar{p} - q\bar{\omega}i}{2\sqrt{i\bar{\omega}} + \bar{p} + q\bar{\omega}i}\exp(-\sqrt{\frac{\bar{\omega}}{2}}L)| \leq \sqrt{\delta}.$$

Hence

$$|\frac{2\sqrt{i\bar{\omega}} - \bar{p} - q\bar{\omega}i}{2\sqrt{i\bar{\omega}} + \bar{p} + q\bar{\omega}i}|\exp(-\sqrt{\frac{\bar{\omega}}{2}}L) \leq \sqrt{\delta}.$$

Therefore

$$|\frac{2\sqrt{i\bar{\omega}} - \bar{p} - q\bar{\omega}i}{2\sqrt{i\bar{\omega}} + \bar{p} + q\bar{\omega}i}| \leq \sqrt{\delta}\exp(L\sqrt{\frac{\bar{\omega}}{2}}).$$

This means $\bar{p}_1 + q_1\bar{\omega}i \in \mathcal{D}(2\sqrt{i\bar{\omega}}, \sqrt{\delta}\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$.

Similarly, we have also $\bar{p}_2 + q_2\bar{\omega}i \in \mathcal{D}(2\sqrt{i\bar{\omega}}, \sqrt{\delta}\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$.

If $\sqrt{\delta}\exp(L\sqrt{\frac{\bar{\omega}}{2}}) < 1$, using Lemma 2.1 in [1], we can see that $\mathcal{D}(2\sqrt{i\bar{\omega}}, \sqrt{\delta}\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$ is convex. Thus for $\theta \in [0,1]$, we have $\theta(\bar{p}_1,q_2)+(1-\theta)(\bar{p}_2,q_2) \in D^L_\delta$.

If $\sqrt{\delta}\exp(L\sqrt{\frac{\bar{w}}{2}}) \geq 1$, using Lemma 2.1 in [1], we can see that for $\bar{p}_1,\ \bar{p}_2,\ q_1,\ q_2 \geq 0,\ \theta \in [0,1]$, we have $\theta(\bar{p}_1 + q_1\omega i) + (1-\theta)(\bar{p}_2 + q_2\bar{\omega}i) \in \mathcal{D}(2\sqrt{i\bar{\omega}}, \sqrt{\delta}\exp(L\sqrt{\frac{\bar{\omega}}{2}}))$. Thus for $\theta \in [0,1]$, we have $\theta(\bar{p}_1,q_1) + (1-\theta)(\bar{p}_2,q_2) \in D^L_\delta$.

Therefore D^L_δ is convex.

Suppose that $(\bar{p}^*, q^*, h_L(\bar{p}^*,q^*))$ is a strictly local minimum of $h_L(\bar{p},q)$, we prove that it is a global minimum of $h_L(\bar{p},q)$. Suppose the contrary that there exists $(\bar{p}^{**}, q^{**}, h_L(\bar{p}^{**}))$ such that $h_L(\bar{p}^*,q^*) \geq h_L(\bar{p}^{**},q^{**})$. Then there exists a convex neighborhood U of $(\bar{p}^*,q^*)$, such that $\forall\ s \in U,\ s \neq (\bar{p}^*,q^*)$ and $h_L(s) > h_L(\bar{p}^*,q^*)$. Since $(\bar{p}^{**},q^{**}) \in D^L_{h_L(\bar{p}^{**},q^{**})} \subset D^L_{h_L(\bar{p}^*,q^*)}$,we have that $\forall\theta \in [0,1]$, $\theta(\bar{p}^*, q^*) + (1-\theta)(\bar{p}^{**}, q^{**}) \in D^L_{h_L(\bar{p}^*,q^*)}$. For θ small enough, we have that $\theta(\bar{p}^{**}, q^{**}) + (1-\theta)(\bar{p}^*, q^*) \in U$. This is a contradiction.

Thus $(\bar{p}^*, q^*)$ is the unique solution of (2.2.4).

$\blacksquare$

Proof of theorem 2.2.1 We put $q' = 4q$, and we have that

$$\begin{aligned}\bar\rho(\bar\omega,\bar p,q,L) &= \left|\frac{2\sqrt{i\bar\omega}-\bar p-q'\bar\omega i}{2\sqrt{i\bar\omega}+\bar p+q'\bar\omega i}\exp(-\sqrt{i\bar\omega}L)\right|^2\\ &= \frac{(\sqrt{2\bar\omega}-\bar p)^2+(\sqrt{2\bar\omega}-q'\bar\omega)^2}{(\sqrt{2\bar\omega}+\bar p)^2+(\sqrt{2\bar\omega}+q'\bar\omega)^2}\exp(-\sqrt{2\bar\omega}L)\\ &= \frac{4\bar\omega-2\sqrt{2}\bar p\sqrt{\bar\omega}-2\sqrt{2}q'\sqrt{\bar\omega^3}+q'^2\bar\omega^2+\bar p^2}{4\bar\omega+2\sqrt{2}\bar p\sqrt{\bar\omega}+2\sqrt{2}q'\sqrt{\bar\omega^3}+q'^2\bar\omega^2+\bar p^2}\exp(-\sqrt{2\bar\omega}L).\end{aligned}$$

Step 1: We consider the behavior of the function $\bar\rho$ with some particular values of $\bar p$ and q.

Suppose that $\bar p = C_pL^{-\gamma_p}$ and $q' = C_q^{\gamma_q}$, $\gamma_p < \gamma_q < 1$, $\gamma_p + \gamma_q \le 1$.

We have that

$$\bar\rho_{\bar\omega}(\bar\omega,\bar p,q,L) = -\frac{\sqrt2}{2}\frac{\exp(-\sqrt{2\bar\omega}L)}{(4\bar\omega+2\sqrt{2}\bar p\sqrt{\bar\omega}+2\sqrt{2}q'\sqrt{\bar\omega^3}+q'^2\bar\omega^2+\bar p^2)^2}\times$$
$$\times(-16\bar p\bar\omega+L\bar p^4+16L\bar\omega^2+4\bar p^3+2Lq'^2p^2\bar\omega^2+16\bar\omega^2q'-12\bar pq'^2\bar\omega^2+$$
$$+12q'\bar p^2\bar\omega-4q'^3\bar\omega^3+Lq'^4\bar\omega^4-16L\bar\omega^2\bar pq').$$

We put

$$\begin{aligned}G(\bar\omega) &= -16\bar p\bar\omega+L\bar p^4+16L\bar\omega^2+4\bar p^3+2Lq'^2\bar p^2\bar\omega^2+16\bar\omega^2q'-\\ &\quad -12\bar pq'^2\bar\omega^2+12q'\bar p^2\bar\omega-4q'^3\bar\omega^3+Lq'^4\bar\omega^4-16L\bar\omega^2\bar pq'\\ &= Lq'^4\bar\omega^4-4q'^3\bar\omega^3+(16L+2Lq'^2\bar p^2+16q'-12\bar pq'^2-16L\bar pq')\bar\omega^2\\ &\quad +(12q'\bar p^2-16\bar p)\bar\omega+L\bar p^4+4\bar p^3.\end{aligned}$$

In order to consider the behavior of $\bar\rho$, we will consider the sign of G.

Consider $\bar\omega$ of the form $C_\omega L^{-\gamma}$. We have the following remarks.

Remark 1: If $1+\gamma_q > \gamma > 2\gamma_q$, then $G(L^{-\gamma}) < 0$ for L small enough.

We have that $4q'^3\bar\omega^3 = 4C_\omega^3C_q^3L^{3\gamma_q-3\gamma}$, thus the order of L in $4q'^3\bar\omega^3$ is $3\gamma_q - 3\gamma$.

We have $Lq'^4\bar\omega^4 = C_\omega^4C_q^4L^{1+4\gamma_q-4\gamma}$; and $1+4\gamma_q-4\gamma > 3\gamma_q-3\gamma$ since $1+\gamma_q > \gamma$.

We have $16L\bar\omega^2 = 16C_\omega^2L^{1-2\gamma}$; and $1-2\gamma > 3\gamma_q-3\gamma$ since $\gamma > 2\gamma_q > 3\gamma_q-1$.

We have $2Lq'^2\bar p^2\bar\omega^2 = 2C_\omega^2C_q^2C_p^2L^{1+2\gamma_q-2\gamma_p-2\gamma}$; and $1+2\gamma_q-2\gamma_p-2\gamma >$

$3\gamma_q - 3\gamma$ since $\gamma > 2\gamma_q > \gamma_q + 2\gamma_p - 1$.

We have $16q'\bar{\omega}^2 = 16C_\omega^2 C_q L^{\gamma_q - 2\gamma}$; and $\gamma_q - 2\gamma > 3\gamma_q - 3\gamma$ since $\gamma > 2\gamma_q$.

We have $12q'p^2\bar{\omega} = 12C_\omega C_q C_p^2 L^{\gamma_q - 2\gamma_p - \gamma}$; and $\gamma_q - 2\gamma_p - \gamma > 3\gamma_q - 3\gamma$ since $\gamma > 2\gamma_q > \gamma_p + \gamma_q$.

We have $L\bar{p}^4 = C_p^4 L^{1-4\gamma_p}$; and $1 - 4\gamma_p > 3\gamma_q - 3\gamma$ since $1 + 3\gamma > 3\gamma_q + 4\gamma_p$.

We have $4\bar{p}^3 = 4C_p^3 L^{-3\gamma_p}$; and $-3\gamma_p > 3\gamma_q - 3\gamma$ since $\gamma > \gamma_p + \gamma_q$.

Thus, among the coefficients of G, the order of L in $-4q'^3 3\bar{\omega}^3$ is smaller than the orders of L in other positive coeficients. This means that for L small enough, we have that $G(L^{-\gamma}) < 0$.

Remark 2: If $\gamma_p + \gamma_q > \gamma > 2\gamma_p$, then $G(L^{-\gamma}) < 0$ for L small enough.

We have that $16\bar{p}\bar{\omega} = 16C_\omega C_p L^{-\gamma-\gamma_p}$, thus the order of L in $16\bar{p}\bar{\omega}$ is $-\gamma - \gamma_p$.

We have $Lq'^4\bar{\omega}^4 = C_\omega^4 C_q^4 L^{1+4\gamma_q - 4\gamma}$; and $1 + 4\gamma_q - 4\gamma > -\gamma - \gamma_p$ since $1 + 4\gamma_q + \gamma_p > 3\gamma$.

We have $16L\bar{\omega}^2 = 16C_\omega^2 L^{1-2\gamma}$; and $1 - 2\gamma > -\gamma - \gamma_p$ since $1 + \gamma_p > \gamma$.

We have $2Lq'^2\bar{p}^2\bar{\omega}^2 = 2\nu C_\omega^2 C_q^2 C_p^2 L^{1+2\gamma_q - 2\gamma_p - 2\gamma}$; and $1 + 2\gamma_q - 2\gamma_p - 2\gamma > -\gamma - \gamma_p$ since $1 + 2\gamma_q - \gamma_p > \gamma$.

We have $16q'\bar{\omega}^2 = 16C_\omega^2 C_q \gamma^{\gamma_q - 2\gamma}$; and $\gamma_q - 2\gamma > -\gamma - \gamma_p$ since $\gamma_p + \gamma_q > \gamma$.

We have $12q'\bar{p}^2\bar{\omega} = 12C_\omega C_q C_p^2 \Delta^{\gamma_q - 2\gamma_p - \gamma}$; and $\gamma_q - 2\gamma_p - \gamma > -\gamma - \gamma_p$ since $\gamma_q > \gamma_p$.

We have $L\bar{p}^4 = \nu^{-1} C_p L^{1-4\gamma_p}$; and $1 - 4\gamma_p > -\gamma - \gamma_p$ since $\gamma > 3\gamma_p - 1$.

We have $4\bar{p}^3 = 4C_p^3 L^{-3\gamma_p}$; and $-3\gamma_p > -\gamma - \gamma_p$ since $\gamma > 2\gamma_p$.

Thus, among the coefficients of G, the order of L in $16\bar{p}\bar{\omega}$ is smaller than the orders of L in other positive coeficients. This means that for L small enough, we have that $G(L^{-\gamma}) < 0$.

Remark 3: If $\gamma_p + \gamma_q < \gamma < 2\gamma_q$, then $G(L^{-\gamma}) > 0$ for L small enough.

The order of L in $16q'\bar{\omega}^2$ is $\gamma_q - 2\gamma$.

The order of L in $4q'^3\bar{\omega}^3$ is $3\gamma_q - 3\gamma$; and $3\gamma_q - 3\gamma > \gamma_q - 2\gamma$ since $2\gamma_q > \gamma$.

The order of L in $12\bar{p}q'^2\bar{\omega}^2$ is $-\gamma_p + 2\gamma_q - 2\gamma$; and $-\gamma_p + 2\gamma_q - 2\gamma > \gamma_q - 2\gamma$ since $\gamma_q > \gamma_p$.

The order of L in $16L\bar{p}q'\bar{\omega}^2$ is $1 + \gamma_q - \gamma_p - 2\gamma$; and $1 + \gamma_q - \gamma_p - 2\gamma > \gamma_q - 2\gamma$ since $1 > \gamma_p$.

The order of L in $16\bar{p}\bar{\omega}$ is $-\gamma_p - \gamma$; and $-\gamma_p - \gamma > \gamma_q - 2\gamma$ since $\gamma > \gamma_p + \gamma_q$.

Thus, among the coefficients of G, the order of L in $16q'\bar{\omega}^2$ is smaller than the orders of L in other negative coeficients. This means that for L small enough, we have that $G(L^{-\gamma}) > 0$.

Combining results 1,2,3 and the facts that $G(0) > 0$ and $\lim_{\omega\to\infty} G(\omega) = +\infty$ we can conclude that G has four positive solutions $\bar{\omega}_1 < \bar{\omega}_2 < \bar{\omega}_3 < \bar{\omega}_4$.

We can see that $\bar{\omega}_2$ and $\bar{\omega}_4$ are the two maximum values of ρ.

We compute $\bar{\omega}_4$. From $G(\bar{\omega}_4) = 0$, we can see that

$$4q'^3\bar{\omega}_4^3 \sim Lq'^4\bar{\omega}_4^4.$$

Hence

$$\bar{\omega}_4 \sim \frac{4}{Lq'}.$$

We compute $\bar{\omega}_2$. From $G(\bar{\omega}_2) = 0$, we can see that

$$16q'\bar{\omega}_2^2 \sim 16\bar{p}\bar{\omega}_2.$$

Thus

$$\bar{\omega}_2 \sim \frac{\bar{p}}{q'}.$$

Since L is small and $\bar{\omega}_{max}$ is fixed, we have that $\bar{\omega}_4(p) > \bar{\omega}_{\max}(\bar{p})$. Hence

$$\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega}, \bar{p}, L) = \{\bar{\rho}(\bar{\omega}_{\min}, \bar{p}, q, L), \bar{\rho}(\bar{\omega}_2, \bar{p}, q, L), \bar{\rho}(\bar{\omega}_{\max}, \bar{p}, q, L)\}.$$

Step 2: We find an approximated solution (p_*, q_*) satisfying the assumptions of (p, q) of the equation $\bar{\rho}(\bar{\omega}_{min}, \bar{p}, q, L) = \bar{\rho}(\bar{\omega}_2, \bar{p}, q, L) = \bar{\rho}(\bar{\omega}_{max}, p, q, L)$.

We have the extension of $\bar{\rho}(\bar{\omega}_{\min}, p, q, L)$

$$\begin{aligned}
\bar{\rho}(\omega_{min}, p, q, L) &= \frac{4\bar{\omega}_{min} - 2\sqrt{2}\bar{p}\sqrt{\bar{\omega}_{min}} - 2\sqrt{2}q'\sqrt{(\bar{\omega}_{min})^3} + q'^2(\bar{\omega}_{min})^2 + \bar{p}^2}{4\bar{\omega}_{min} + 2\sqrt{2}\bar{p}\sqrt{\bar{\omega}_{min}} + 2\sqrt{2}q'\sqrt{(\bar{\omega}_{min})^3} + q'^2(\bar{\omega}_{min})^2 + \bar{p}^2} \times \\
&\quad \times \exp(-\sqrt{2\bar{\omega}_{min}}L) \\
&= \frac{\frac{4\bar{\omega}_{min}}{\bar{p}^2} - \frac{2\sqrt{2}\sqrt{\bar{\omega}_{min}}}{p} - \frac{2\sqrt{2}q'\sqrt{\bar{\omega}_{min}^3}}{\bar{p}^2} + \frac{q'^2}{\bar{p}^2}\bar{\omega}_{min}^2 + 1}{\frac{4\bar{\omega}_{min}}{\bar{p}^2} + \frac{2\sqrt{2}\sqrt{\bar{\omega}_{min}}}{\bar{p}} + \frac{2\sqrt{2}q'\sqrt{\bar{\omega}_{\min}^3}}{\bar{p}^2} + \frac{q'^2}{\bar{p}^2}\bar{\omega}_{min}^2 + 1} \exp(-\sqrt{2\bar{\omega}_{min}}L) \\
&\backsim (1 - \frac{4\sqrt{2\bar{\omega}_{min}}}{\bar{p}})(1 - \sqrt{2\bar{\omega}_{min}}L) \\
&\backsim 1 - \frac{4\sqrt{2\bar{\omega}_{min}}}{\bar{p}}.
\end{aligned}$$

We have the extension of $\bar\rho(\bar\omega_2,\bar p,q,L)$

$$\begin{aligned}
\bar\rho(\bar\omega_2,p,q,L) &= \frac{4\bar\omega_2-2\sqrt{2}\bar p\sqrt{\bar\omega_2}-2\sqrt{2}q'\sqrt{\bar\omega_2}^3+q'^2(\bar\omega_2)^2+\bar p^2}{4\bar\omega_2+2\sqrt{2}\bar p\sqrt{\bar\omega_2}+2\sqrt{2}q'\sqrt{\bar\omega_2}^3+q'^2(\bar\omega_2)^2+p^2}\exp(-\sqrt{2\bar\omega_2}L)\\
&= \frac{1-q'\sqrt{\frac{\bar\omega_2}{2}}+\frac{q'^2\bar\omega_2}{4}-\sqrt{\frac{1}{2\bar\omega_2}}\bar p+\frac{\bar p^2}{4\bar\omega_2}}{1+q'\sqrt{\frac{\bar\omega_2}{2}}+\frac{q'^2\bar\omega_2}{4}+\sqrt{\frac{1}{2\bar\omega_2}}\bar p+\frac{\bar p^2}{4\bar\omega_2}}\exp(-\sqrt{2\bar\omega_2}L)\\
&\backsim 1-q'\sqrt{2\bar\omega_2}-\sqrt{\frac{2}{\bar\omega_2}}\bar p\\
&\backsim 1-2\sqrt{2\bar p q'}.
\end{aligned}$$

We have the extension of $\bar\rho(\bar\omega_{\max},\bar p,q,L)$

$$\begin{aligned}
\bar\rho(\bar\omega_{max},\bar p,q,L) &= \frac{4\bar\omega_{max}-2\sqrt{2}\bar p\sqrt{\bar\omega_{max}}-2\sqrt{2}q'\sqrt{(\bar\omega_{max})^3}+q'^2(\bar\omega_{max})^2+\bar p^2}{4\bar\omega_{max}+2\sqrt{2}\bar p\sqrt{\bar\omega_{max}}+2\sqrt{2}q'\sqrt{(\bar\omega_{max})^3}+q'^2(\bar\omega_{max})^2+\bar p^2}\times\\
&\quad\times\exp(-\sqrt{2\bar\omega_{max}}L).
\end{aligned}$$

We have that

$$\bar\rho(\bar\omega_{max},\bar p,q,L)\backsim 1-\sqrt{2}q'^{-1}(\bar\omega_{max}\nu)^{-\frac{1}{2}}.$$

We need to solve the following equations

$$\frac{4\sqrt{2\bar\omega_{min}}}{p}=2\sqrt{2\bar p q'}=\sqrt{2}q'^{-1}(\bar\omega_{max}\nu)^{-\frac{1}{2}}.$$

Thus

$$16\bar\omega_{min}=4\bar p^3q'=q'^{-2}\bar p^2(\bar\omega_{max})^{-1}.$$

We get that

$$\bar p^3q'=4\bar\omega_{min},$$

and

$$\bar p q'^3=\frac{1}{4}(\bar\omega_{max})^{-1}.$$

Hence

$$\bar p=2^{\frac{1}{2}}(\bar\omega_{min})^{\frac{3}{8}}(\bar\omega_{max})^{\frac{1}{8}}=:\bar p_*,$$

$$q = 2^{-\frac{5}{2}}(\bar{\omega}_{min})^{-\frac{1}{8}}(\bar{\omega}_{max})^{-\frac{3}{8}} =: q_*.$$

Step 3: We prove that $(\bar{p}, q')$ is a strictly local minimum of $\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]}$ $\bar{\rho}(\bar{\omega}, \bar{p}, q, L)$, then according to Lemma 2.2.1, $(\bar{p}, q)$ is also a global minimum.

The pair $(\bar{p}, q)$ is a strictly local minimum if there exists no variation $(\delta\bar{p}, \delta q)$ such that $\bar{\rho}(\bar{\omega}, \bar{p}+\delta p, q+\delta q, L) < \bar{\rho}(\bar{\omega}, \bar{p}, q, L)$ for $\bar{\omega} = \bar{\omega}_{min}, \bar{\omega}_2, \bar{\omega}_{max}$. By the Taylor formula, it suffices to prove that there is no variation $(\delta\bar{p}, \delta q)$, such that $\delta\bar{p}\frac{\partial\bar{\rho}}{\partial\bar{p}}(\bar{\omega}, \bar{p}_*, q_*, L) + \delta q\frac{\partial\bar{\rho}}{\partial q}(\bar{\omega}, \bar{p}_*, q_*, L) > 0$ for $\bar{\omega} = \bar{\omega}_{min}, \bar{\omega}_2, \bar{\omega}_{max}$.

Suppose that there exists $(\delta\bar{p}, \delta q)$ such that $\delta\bar{p}\frac{\partial\bar{\rho}}{\partial\bar{p}}$ $(\bar{\omega}, \bar{p}_*, q_*, L)$ + δq $\frac{\partial\bar{\rho}}{\partial q}$ $(\bar{\omega}, \bar{p}_*, q_*, L) > 0$.

We have that

$$\begin{aligned}
\frac{\partial\bar{\rho}}{\partial\bar{p}} &= -\frac{4\exp(-\sqrt{2}\sqrt{\bar{\omega}}L)(4\bar{\omega} - \bar{p}^2 + q^2\bar{\omega}^2 - 2\bar{p}q\bar{\omega})}{(4\bar{\omega} + 2\sqrt{2}\sqrt{\bar{\omega}}\bar{p} + \bar{p}^2 + 2\sqrt{2}\sqrt{\bar{\omega}}q\bar{\omega} + q^2\bar{\omega}^2)^2}, \\
\frac{\partial\bar{\rho}}{\partial q} &= \frac{4\bar{\omega}\exp(-\sqrt{2}\sqrt{\bar{\omega}}L)(-4\bar{\omega} - \bar{p}^2 + q^2\bar{\omega}^2 + 2\bar{p}q\bar{\omega})}{(4\bar{\omega} + 2\sqrt{2}\sqrt{\bar{\omega}}\bar{p} + \bar{p}^2 + 2\sqrt{2}\sqrt{\bar{\omega}}q\bar{\omega} + q^2\omega^2)^2}.
\end{aligned}$$

We have $\delta\bar{p}\frac{\partial\bar{\rho}}{\partial\bar{p}}(\bar{\omega}_{min}, \bar{p}_*, q_*, L)$ $+\delta q\frac{\partial\bar{\rho}}{\partial q}(\bar{\omega}_{min}, \bar{p}_*, q_*, L) \backsim M_1\bar{p}_*^2(\delta p{-}\delta q\bar{\omega}_{min}) >$ 0. Hence $\delta\bar{p} - \delta q\bar{\omega}_{min} > 0$. Moreover, $\delta p\frac{\partial\rho}{\partial\bar{p}}$ $(\bar{\omega}_2, \bar{p}_*, q_*, L)+$ $\delta q\frac{\partial\rho}{\partial q}(\bar{\omega}_2, \bar{p}_*, q_*, L)$ $\backsim M_2$ $(-\delta p4\sqrt{2}$ $(\bar{\omega}_{min})^{-\frac{1}{8}}$ $(\bar{\omega}_{max})^{\frac{3}{8}}$ $+\delta q64\sqrt{2}$ $(\bar{\omega}_{min})^{\frac{9}{8}}$ $(\bar{\omega}_{max})^{\frac{11}{8}}) > 0$. Thus $-\delta p4\sqrt{2}$ $(\bar{\omega}_{min})^{-\frac{3}{8}}$ $(\bar{\omega}_{max})^{\frac{7}{8}}$ $+\delta q\frac{64\sqrt{2}}{3}$ $(\bar{\omega}_{min})^{-\frac{1}{8}}$ $(\bar{\omega}_{max})^{-\frac{3}{8}} > 0$. Hence $\delta\bar{p}$, δq > 0. We have also that $\delta\bar{p}$ $\frac{\partial\rho}{\partial\bar{p}}(\bar{\omega}_{max}$ $, \bar{p}_*, q_*, L)+$ $\delta q\frac{\partial\rho}{\partial q}(\bar{\omega}_{max}, \bar{p}_*, q_*, L) \backsim M_3\bar{p}_*^2$ $(-\delta p4\sqrt{2}(\bar{\omega}_{min})^{-\frac{3}{8}}$ $(\bar{\omega}_{max})^{\frac{7}{8}}$ $-\delta q\frac{32\sqrt{2}}{3}$ $(\bar{\omega}_{min})^{\frac{1}{8}}$ $(\bar{\omega}_{max})^{\frac{19}{8}}) > 0$. This is a contradiction with the fact that $\delta\bar{p}$ and $\delta q > 0$.

Using Lemma 2.2.1 and the same argument as in the previous section we can easily see that the solution $(\bar{p}_*, q_*)$ that $(\bar{p}, q)$ approximates is a local minimum of $\bar{\rho}(\bar{\omega}, \bar{p}, q, L)$. Thus

$$\bar{p}_* = 2^{\frac{1}{2}}(\bar{\omega}_{min})^{\frac{3}{8}}(\bar{\omega}_{max})^{\frac{1}{8}},$$

$$q_* = 2^{-\frac{5}{2}}(\bar{\omega}_{min})^{-\frac{1}{8}}(\bar{\omega}_{max})^{-\frac{3}{8}},$$

$$\min_{\bar{p},q\geq 0}\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]}\bar{\rho}(\bar{\omega}, \bar{p}, q, L) \backsim 1 - 4(\bar{\omega}_{min})^{\frac{1}{8}}(\bar{\omega}_{max})^{-\frac{1}{8}}.$$

Proof of theorem 2.2.2

Case 1: $L = C_1\Delta x,\ \Delta t = C_2\Delta x,\ \bar\omega \in [\frac{\pi}{2T\nu}, \frac{\pi}{C_2\nu}\Delta x^{-1}]$.

Put $q' = 4q$, we suppose that $\bar p = C_p\Delta x^{-\gamma_p}$, $q' = C_q\Delta x^{\gamma_q}$, and $1 > \gamma_q > \gamma_p$ and $\gamma_p + \gamma_p \leq 1$.

$$\begin{aligned}
\bar\rho(\bar\omega, \bar p, q, L) &= |\frac{2\sqrt{i\bar\omega} - \bar p - q'\bar\omega i}{2\sqrt{i\bar\omega} + \bar p + q'\bar\omega i}\exp(-\sqrt{i\bar\omega}L)|^2 \\
&= \frac{(\sqrt{2\bar\omega} - \bar p)^2 + (\sqrt{2\bar\omega} - q'\bar\omega)^2}{(\sqrt{2\bar\omega} + \bar p)^2 + (\sqrt{2\bar\omega} + q'\bar\omega)^2}\exp(-\sqrt{2\bar\omega}L) \\
&= \frac{4\bar\omega - 2\sqrt{2}\bar p\sqrt{\bar\omega} - 2\sqrt{2}q'\sqrt{(\bar\omega)^3} + q'^2(\bar\omega)^2 + \bar p^2}{4\bar\omega + 2\sqrt{2}\bar p\sqrt{\bar\omega} + 2\sqrt{2}q'\sqrt{(\bar\omega)^3} + q'^2(\bar\omega)^2 + \bar p^2}\exp(-\sqrt{2\bar\omega}L).
\end{aligned}$$

Hence

$$\partial_{\bar\omega}\bar\rho(\bar\omega, \bar p, q, L) = -\frac{\sqrt{2}}{2}\frac{\exp(-\sqrt{2\bar\omega}L)}{(4\bar\omega + 2\sqrt{2}\bar p\sqrt{\bar\omega} + 2\sqrt{2}q'\sqrt{(\bar\omega)^3} + q'^2(\bar\omega)^2 + \bar p^2)^2}\times$$

$$\times(-16\bar p\bar\omega + L\bar p^4 + 16L(\bar\omega)^2 + +4\bar p^3 + 2Lq'^2\bar p^2(\bar\omega)^2 + 16(\bar\omega)^2q' -$$
$$-12\bar p q'^2(\bar\omega)^2 + 12q'\bar p^2\bar\omega - 4q'^3(\bar\omega)^3 + L\bar p^4(\bar\omega)^4 - 16L(\bar\omega)^2\bar p q').$$

Using the same argument as in the previous theorem, we can see that

$$\bar p_* = 2^{\frac{1}{2}}(\frac{\pi^4\nu^{-4}}{8T^3C_2})^{\frac{1}{8}}\Delta x^{-\frac{1}{8}},$$

$$q_* = 2^{-\frac{5}{2}}(\frac{\pi^4\nu^4}{2TC_2^3})^{-\frac{1}{8}}\Delta x^{\frac{3}{8}},$$

$$\min_{\bar p, q\geq 0}\max_{\bar\omega\in[\frac{\pi}{2T\nu}, \frac{\pi}{\Delta t\nu}]}\rho(\bar\omega, \bar p, q, L) = 1 - 2^{\frac{7}{4}}(\frac{C_2}{2T})^{\frac{1}{8}}\Delta x^{\frac{1}{8}} + O(\Delta x^{\frac{1}{4}}).$$

Case 2: $L = C_1\Delta x,\ \Delta t = C_2\Delta x^2,\ \bar\omega \in [\frac{\pi}{2T\nu}, \frac{\pi}{C_2\nu}\Delta x^{-2}]$.

We put $q' = 4q$ and suppose that $\bar p = C_p\Delta x^{-\gamma_p}$, $q' = C_q\Delta x^{\gamma_q}$ and $1 > \gamma_q > \gamma_p$ and $\gamma_p + \gamma_p \leq 1$. We suppose that $\gamma_q - \gamma_p < 1 - \frac{\gamma_p + \gamma_q}{2}$.

Using the same argument as in the previous case, we can see that ρ has two maximum values at $\bar\omega_2$ and $\bar\omega_4$.

$$\bar\omega_4 \sim \frac{4}{C_1C_q}\Delta x^{-1-\gamma_q},$$

and

$$\bar{\omega}_2 \sim \frac{C_p}{C_q}\Delta^{-\gamma_p-\gamma_q}.$$

In this case, we can see that $\bar{\omega}_2 < \bar{\omega}_4 < \bar{\omega}_{\max}$ for Δx small enough. Thus

$$\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega},\bar{p},q,L) = \max\{\bar{\rho}(\bar{\omega}_2,\bar{p},q,L), \bar{\rho}(\bar{\omega}_4,\bar{p},q,L), \bar{\rho}(\bar{\omega}_{min},\bar{p},q,L)\}.$$

Next, we will find a solution of $\bar{\rho}(\bar{\omega}_2,\bar{p},q,L) = \bar{\rho}(\bar{\omega}_4,\bar{p},q,L) = \bar{\rho}(\bar{\omega}_{min},\bar{p},q,L)$ asymptotically.

We have the extension of $\bar{\rho}(\bar{\omega}_{\min},\bar{p},q,L)$

$$\bar{\rho}(\bar{\omega}_{\min},\bar{p},q,L) = 1 - \frac{4\sqrt{2\bar{\omega}_{min}}}{C_p}\Delta x^{\gamma_p} + O(\Delta x^{2\gamma_p}).$$

We have the extension of $\rho(\bar{\omega}_2,\bar{p},q,L)$

$$\bar{\rho}(\bar{\omega}_2,\bar{p},q,L) = 1 - 2\sqrt{2C_pC_q}\Delta x^{\frac{\gamma_q-\gamma_p}{2}} + O(\Delta x^{\gamma_q-\gamma_p}).$$

We have the extension of $\rho(\bar{\omega}_4,\bar{p},q,L)$

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_4,\bar{p},q,L) &= \frac{4\bar{\omega}_4 - 2\sqrt{2}p\sqrt{\bar{\omega}_4} - 2\sqrt{2}q'\sqrt{\bar{\omega}_4^3} + q'^2\bar{\omega}_4^2 + \bar{p}^2}{4\bar{\omega}_4 + 2\sqrt{2}\bar{p}\sqrt{\bar{\omega}_4} + 2\sqrt{2}q'\sqrt{\bar{\omega}_4^3} + q'^2\bar{\omega}_4^2 + \bar{p}^2}\exp(-\sqrt{2\bar{\omega}_4}L) \\
&= \frac{4q'^{-2}\bar{\omega}_4^{-1} - 2\sqrt{2}\bar{p}q'^{-2}\sqrt{\bar{\omega}_4^{-3}} - 2\sqrt{2}q'^{-1}\sqrt{\bar{\omega}_4^{-1}} + 1 + \bar{p}^2q'^{-2}\bar{\omega}_4^{-2}}{4q'^{-2}\bar{\omega}_4^{-1} + 2\sqrt{2}\bar{p}q'^{-2}\sqrt{\bar{\omega}_4^{-3}} + 2\sqrt{2}q'^{-1}\sqrt{\bar{\omega}_4^{-1}} + 1 + \bar{p}^2q'^{-2}\bar{\omega}_4^{-2}} \times \\
&\quad \times \exp(-\sqrt{2\bar{\omega}_4}L) \\
&= (1 - 4\sqrt{2}q^{-1}\sqrt{\bar{\omega}_4^{-1}} + O(\Delta x^{1-\gamma_q}))(1 - \sqrt{2\bar{\omega}_4}L + O(\Delta x^{1-\gamma_q})) \\
&= (1 - 4\sqrt{2}C_q^{-1}\sqrt{4^{-1}C_1\nu^{-1}C_q}\Delta x^{\frac{1-\gamma_q}{2}} + O(\Delta x^{1-\gamma_q}))(1 - \\
&\quad \sqrt{8C_1^{-1}C_q^{-1}}C_1\nu^{-1}\Delta x^{\frac{1-\gamma_q}{2}} + O(\Delta x^{1-\gamma_q})) \\
&= 1 - 4\sqrt{2}\sqrt{C_1\nu^{-1}C_q^{-1}}\Delta x^{\frac{1-\gamma_q}{2}} + O(\Delta x^{1-\gamma_q}).
\end{aligned}$$

We will solve the equation

$$\frac{4\sqrt{2\bar{\omega}_{min}}}{C_p}\Delta x^{\gamma_p} = 2\sqrt{2C_pC_q}\Delta x^{\frac{\gamma_q-\gamma_p}{2}} = 4\sqrt{2}\sqrt{C_1\nu^{-1}C_q^{-1}}\Delta x^{\frac{1-\gamma_q}{2}}.$$

Hence

$$\gamma_p = \frac{\gamma_q - \gamma_p}{2} = \frac{1-\gamma_q}{2}.$$

From

$$\gamma_p = \frac{\gamma_q - \gamma_p}{2},$$

we have that

$$\gamma_q = 3\gamma_p.$$

From

$$\gamma_p = \frac{1-\gamma_q}{2},$$

we have that

$$2\gamma_p = 1 - \gamma_q = 1 - 3\gamma_p.$$

Thus

$$\gamma_p = \frac{1}{5}.$$

Hence

$$\gamma_q = \frac{3}{5}.$$

We have the equation

$$\frac{4\sqrt{2\bar{\omega}_{min}}}{C_p} = 2\sqrt{2C_pC_q} = 4\sqrt{2}\sqrt{C_1\nu^{-1}C_q^{-1}},$$

or

$$\frac{2\sqrt{\bar{\omega}_{min}}}{C_p} = \sqrt{C_pC_q} = 2\sqrt{C_1\nu^{-1}C_q^{-1}},$$

or

$$\frac{4\bar{\omega}_{min}}{C_p^2} = C_pC_q = 4C_1\nu^{-1}C_q^{-1}.$$

Thus

$$C_pC_q^2 = 4C_1\nu^{-1},$$

and

$$C_p^3C_q = 4\bar{\omega}_{min}.$$

We have that

$$C_p^3C_q^6 = 64(C_1\nu^{-1})^3.$$

Thus

$$C_q^5 = 16(C_1\nu^{-1})^3\bar{\omega}_{min}^{-1}.$$

Hence

$$C_q = (16(C_1\nu^{-1})^3\bar{\omega}_{min}^{-1})^{\frac{1}{5}}.$$

From

$$C_pC_q^2 = 4\omega_{min}\nu,$$

we have that

$$C_p(16(C_1\nu^{-1})^3\bar{\omega}_{min}^{-1})^{\frac{2}{5}} = 4(C_1\nu^{-1}).$$

Thus

$$C_p = 4C_1\nu^{-1}(256^{-1}\bar{\omega}_{min}^2 2(C_1\nu^{-1})^{-6})^{\frac{1}{5}}.$$

Hence

$$C_p = (4\bar{\omega}_{min}^2\nu^2(C_1\nu^{-1})^{-1})^{\frac{1}{5}}.$$

Thus

$$\bar{p} = (4\bar{\omega}_{min}^2(C_1)^{-1})^{\frac{1}{5}}\Delta x^{\frac{1}{5}},$$

$$q = \frac{1}{4}(16(C_1\nu^{-1})^3\bar{\omega}_{min}^{-1}\nu^{-1})^{\frac{1}{5}}\Delta x^{\frac{3}{5}}.$$

Using Lemma 2.2.1 and the same argument as in the previous section we can easily see that the solution $(\bar{p}_*, q_*)$ that (p, q) approximates is a local minimum of $\bar{\rho}(\bar{\omega}, \bar{p}, q, L)$. Thus

$$\bar{p}_* = (4\bar{\omega}_{min}^2\nu^2(C_1\nu^{-1})^{-1})^{\frac{1}{5}}\Delta x^{\frac{1}{5}},$$

$$q_* = \frac{1}{4}(16(C_1\nu^{-1})^3\bar{\omega}_{min}^{-1}\nu^{-1})^{\frac{1}{5}}\Delta x^{\frac{3}{5}},$$

$$\min_{\bar{p},q\geq 0}\max_{\bar{\omega}\in[\frac{\pi}{2T\nu},\frac{\pi}{\Delta t\nu}]}\bar{\rho}(\bar{\omega},\bar{p},q,L) = 1 - 2\sqrt{2C_pC_q}\Delta x^{\frac{\gamma_q-\gamma_p}{2}} + O(\Delta x^{\gamma_q-\gamma_p})$$

$$= 1 - 2\sqrt{2(4\omega_{min}^2\nu^2(C_1\nu^{-1})^{-1})^{\frac{1}{5}}(16(C_1\nu^{-1})^3\omega_{min}^{-1}\nu^{-1})^{\frac{1}{5}}}\Delta x^{\frac{1}{5}} + O(\Delta x^{\frac{2}{5}})$$

$$= 1 - 2^{\frac{21}{10}}\omega_{min}^{\frac{1}{10}}\nu^{\frac{1}{10}}(C_1\nu^{-1})^{\frac{1}{5}}\Delta x^{\frac{1}{5}} + O(\Delta x^{\frac{2}{5}})$$

$$= 1 - 2^{\frac{21}{10}}(\frac{\pi\nu}{2T})^{\frac{1}{10}}(C_1\nu^{-1})^{\frac{1}{5}}\Delta x^{\frac{1}{5}} + O(\Delta x^{\frac{2}{5}}).$$

■

2.2.2 Proof of the Theorems in the Nonoverlapping Case

Proof of Theorem 2.2.3

We can see that

$$\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega},\bar{p},q) = \max_{\bar{\omega}_i\in I}\{\bar{\rho}(\bar{\omega}_{min}),\bar{\rho}(\bar{\omega}_{max}),\bar{\rho}(\bar{\omega}_i)\},$$

where $I = \{\bar{\omega}_i | \partial_{\bar{\omega}}\bar{\rho}(\bar{\omega}_i,\bar{p},q) = 0\}$.

Taking the derivative of $\bar{\rho}$ with respect to $\bar{\omega}$, we have that

$$\partial_{\bar{\omega}}\bar{\rho}(\bar{\omega},\bar{p},q) = \frac{2\sqrt{2}(-\bar{p}^3 - 4\bar{\omega}^2 q - 3q\bar{\omega}\bar{p}^2 + q^3\bar{\omega}^3 + 4\bar{\omega}\bar{p} + 3\bar{p}q^3\bar{\omega}^2)}{(4\bar{\omega} + 2\sqrt{2\bar{\omega}}\bar{p} + \bar{p}^2 + 2\sqrt{2\bar{\omega}}q\bar{\omega} + q^2\bar{\omega}^2)^2\sqrt{\bar{\omega}}}.$$

We will try to solve the equation $\partial_{\bar{\omega}}\bar{\rho} = 0$ or the following equation

$$0 = -\bar{p}^3 - 4\bar{\omega}^2 q - 3q\bar{\omega}\bar{p}^2 + q^3\bar{\omega}^3 + 4\bar{\omega}\bar{p} + 3\bar{p}q^2\bar{\omega}^2,$$

or

$$0 = q^3\bar{\omega}^3 + \bar{\omega}^2(3\bar{p}q^2 - 4q) + \bar{\omega}(4\bar{p} - 3q\bar{p}^2) - \bar{p}^3.$$

Suppose that $\bar{p}q$ and q is small, and $\bar{p}$ is large, we can deduce from the above equation that

$$0 = q^3\bar{\omega}^3 - 4q\bar{\omega}^2 + 4\bar{p}\omega - \bar{p}^3.$$

This equation is equivalent to

$$0 = (q\bar{\omega} - \bar{p})(q^2\bar{\omega}^2 + (\bar{p}q - 4)\bar{\omega} + \bar{p}^2).$$

We can see that the first solution is $\bar{\omega}_1 = \frac{\bar{p}}{q}$.

The equation

$$q^2\bar{\omega}^2 + (\bar{p}q - 4)\bar{\omega} + \bar{p}^2 = 0,$$

has the following solutions

$$S_1 = \frac{4 - \bar{p}q + \sqrt{-3\bar{p}^2q^2 - 8\bar{p}q + 16}}{2q^2} \backsim \frac{4}{q^2},$$

and

$$S_2 = \frac{4 - \bar{p}q - \sqrt{-3\bar{p}^2q^2 - 8\bar{p}q + 16}}{2q^2} \backsim -\frac{1}{2}\bar{p}q^{-1}.$$

Thus the second solution of our equation is $\bar{\omega}_2 \backsim 4q^{-2}$.
Therefore

$$\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega},\bar{p},q) = \max\{\bar{\rho}(\bar{\omega}_{min}), \rho(\bar{\omega}_{max}), \bar{\rho}(\bar{\omega}_1), \bar{\rho}(\bar{\omega}_2)\}.$$

For $\bar{\rho}(\bar{\omega}_{min},\bar{p},q)$, we have

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_{min},\bar{p},q) &= \frac{(\sqrt{2\bar{\omega}_{min}}-\bar{p})^2+(\sqrt{2\bar{\omega}_{min}}-q\bar{\omega}_{min})^2}{(\sqrt{2\bar{\omega}_{min}}+\bar{p})^2+(\sqrt{2\bar{\omega}_{min}}+q\bar{\omega}_{min})^2} \\
&= \frac{4\bar{\omega}_{min}-2\sqrt{2\bar{\omega}_{min}}\bar{p}+\bar{p}^2-2q\bar{\omega}_{min}\sqrt{2\bar{\omega}}+q^2\bar{\omega}_{min}^2}{4\bar{\omega}_{min}+2\sqrt{2\bar{\omega}_{min}}\bar{p}+\bar{p}^2+2q\bar{\omega}_{min}\sqrt{2\bar{\omega}}+q^2\bar{\omega}_{min}^2} \\
&\backsim \frac{\bar{p}^2-2\sqrt{2\bar{\omega}_{min}}\bar{p}}{\bar{p}^2+2\sqrt{2\bar{\omega}_{min}}\bar{p}} \backsim 1-\frac{4\sqrt{2\bar{\omega}_{min}}}{\bar{p}}.
\end{aligned}$$

For $\bar{\rho}(\bar{\omega}_1,\bar{p},q)$, we have

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_1,\bar{p},q) &\backsim \frac{(\sqrt{2\bar{p}q^{-1}}-\bar{p})^2+(\sqrt{2\bar{p}q^{-1}}-q\bar{p}q^{-1})^2}{(\sqrt{2\bar{p}q^{-1}}+\bar{p})^2+(\sqrt{2\bar{p}q^{-1}}+q\bar{p}q^{-1})^2} \\
&\backsim \frac{(\sqrt{2\bar{p}q^{-1}}-\bar{p})^2+(\sqrt{2\bar{p}q^{-1}}-\bar{p})^2}{(\sqrt{2\bar{p}q^{-1}}+\bar{p})^2+(\sqrt{2\bar{p}q^{-1}}+\bar{p})^2} \backsim \frac{2\bar{p}q^{-1}+\bar{p}^2-2\sqrt{2\bar{p}q^{-1}\bar{p}^2}}{2\bar{p}q^{-1}+\bar{p}^2+2\sqrt{2\bar{p}q^{-1}\bar{p}^2}} \\
&\backsim 1-2\sqrt{2\bar{p}q}.
\end{aligned}$$

For $\bar{\rho}(\bar{\omega}_2,\bar{p},q)$, we have

$$\bar{\rho}(\bar{\omega}_2,\bar{p},q) \backsim \frac{(\sqrt{8q^{-2}}-\bar{p})^2+(\sqrt{8q^{-2}}-q4q^{-2})^2}{(\sqrt{8q^{-2}}+\bar{p})^2+(\sqrt{8q^{-2}}+q4q^{-2})^2} \backsim \frac{8+(2\sqrt{2}-4)^2}{8+(2\sqrt{2}+4)^2}.$$

Thus

$$\max_{\bar{\omega}\in[\bar{\omega}_{min},\bar{\omega}_{max}]} \bar{\rho}(\bar{\omega},\bar{p},q) = \max\{\bar{\rho}(\bar{\omega}_{max}), 1-\frac{4\sqrt{2\bar{\omega}_{min}}}{\bar{p}}, 1-2\sqrt{2\bar{p}q}\}.$$

We have that

$$\begin{aligned}
\bar{\rho}(\bar{\omega}_{max},\bar{p},q) &= \frac{(\sqrt{2\bar{\omega}_{max}}-\bar{p})^2+(\sqrt{2\bar{\omega}_{max}}-q\bar{\omega}_{max})^2}{(\sqrt{2\bar{\omega}_{max}}+\bar{p})^2+(\sqrt{2\bar{\omega}_{max}}+q\bar{\omega}_{max})^2} \\
&= \frac{4\bar{\omega}_{max}-2\sqrt{2\bar{\omega}_{max}}\bar{p}+\bar{p}^2-2q\bar{\omega}_{max}\sqrt{2\bar{\omega}_{max}}+q^2\bar{\omega}_{max}^2}{4\bar{\omega}_{max}+2\sqrt{2\bar{\omega}_{max}}\bar{p}+\bar{p}^2+2q\bar{\omega}_{max}\sqrt{2\bar{\omega}_{max}}+q^2\bar{\omega}_{max}^2} \\
&\backsim \frac{q^2\bar{\omega}_{max}^2-2\sqrt{2\bar{\omega}_{max}}\bar{\omega}_{max}q}{q^2\bar{\omega}_{max}^2+2\sqrt{2\bar{\omega}_{max}}\bar{\omega}_{max}q} \backsim 1-4\sqrt{2}(\bar{\omega}_{max})^{-\frac{1}{2}}q^{-1}.
\end{aligned}$$

Similar as in the previous sections, we will solve the following equilibrating equation

$$\frac{4\sqrt{2\bar{\omega}_{min}}}{\bar{p}} = 2\sqrt{2\bar{p}q} = 4\sqrt{2}(\bar{\omega}_{max})^{-\frac{1}{2}}q^{-1}.$$

We have that

$$\bar{p}^3 q = 4\bar{\omega}_{min},$$

and

$$\bar{p}q^3 = 4(\bar{\omega}_{max})^{-1}.$$

Therefore

$$\bar{p}q = 2(\bar{\omega}_{min})^{\frac{1}{4}}(\bar{\omega}_{max})^{-\frac{1}{4}}.$$

Hence

$$\bar{p} = \sqrt{2}(\bar{\omega}_{min})^{\frac{3}{8}}(\bar{\omega}_{max})^{\frac{1}{8}} = \sqrt{2}(\frac{\pi}{2T\nu})^{\frac{3}{8}}(\pi\nu^{-1})^{\frac{1}{8}}\Delta t^{-\frac{1}{8}},$$

and

$$q = \sqrt{2}(\omega_{min}\nu)^{-\frac{1}{8}}(\omega_{max}\nu)^{-\frac{3}{8}} = \sqrt{2}(\frac{\pi\nu}{2T})^{-\frac{1}{8}}(\pi\nu)^{-\frac{3}{8}}\Delta t^{\frac{3}{8}}.$$

Using the same argument as in the previous section, we can see that $(p_*, q_*) = (\sqrt{2}(\omega_{min}\nu)^{\frac{3}{8}}(\omega_{max}\nu)^{\frac{1}{8}}, \sqrt{2}(\omega_{min}\nu)^{-\frac{1}{8}}(\omega_{max}\nu)^{-\frac{3}{8}})$ is the unique solution of (2.2.4) and we have that

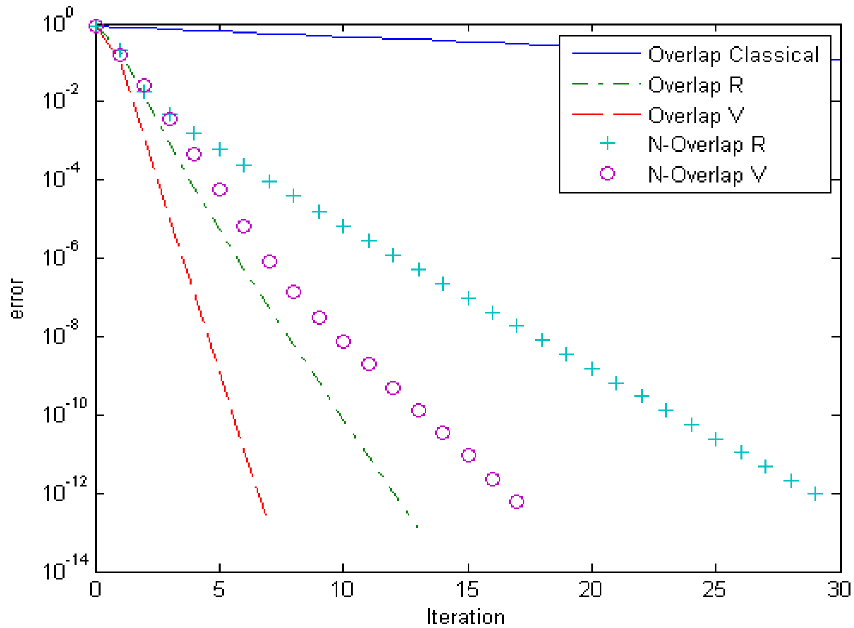
$$\begin{aligned}\min_{p,q\geq 0}\max_{\omega\in[\omega_{min},\omega_{max}]}\bar{\rho}(\omega,p,q) &= \max_{\omega\in[\omega_{min},\omega_{max}]}\bar{\rho}(\omega,p_*,q_*) \backsim 1-4(\omega_{min}\nu)^{\frac{1}{8}}(\omega_{max}\nu)^{-\frac{1}{8}} \\ &\backsim 1-4(\frac{1}{2T})^{\frac{1}{8}}\Delta t^{\frac{1}{8}}.\end{aligned}$$

2.3 Numerical Results

In this section, we perform a series of one dimensional numerical experiments to verify our theoretical results on the optimized parameters for the optimized Schwarz methods obtained in the previous sections. In this set of experiments, we chose for the problem parameters $\nu = 1$, in the domain $[0, 1]$, $T = 2$. We use homogeneous boundary conditions. We discretize the problem with Euler backward scheme and use random initial conditions.

2.3.1 Test 1

First, we would like to compare the behavior of the classical Schwarz method and optimized Schwarz methods with Robin and Ventcell tranmission conditions in both overlapping and nonoverlapping cases. We choose 300 grid points on both the time interval and the space interval, the overlapping length for the overlapping algorithms is 2 grid points. We choose the parameter p for the Robin transmission condition to be our computed optimal p and the parameter (p, q) for the first order transmission condition to be our computed optimal (p, q) and plot the error with respect to the number of interations. We can see that the optimized Schwarz methods converge much faster than the classical one and the optimized Schwarz with the optimal first order transmission condition converges faster than the optimal Robin one as in Figure 2.3.1: The optimized Schwarz methods need only a few iterations to get the errors of 0.01, while classical Schwarz methods need around 100 iterations.



error
Iteration
Overlap Classical
Overlap R
Overlap V
N-Overlap R
N-Overlap V

Figure 2.3.1

2.3.2 Test 2

Now, we would like to test the accuracy of our asymptotics analysis for the optimized Robin parameters. We choose 10 grid points in space and 100 grid points in time and the overlapping length is 2 grid points. On the figures, we let p vary from 0 to 35, the theoretical optimal p is the star $*$ on the curve. The test corresponds to the case $dt = dx^2$. Plotting the errors after 8 interations, we can see that the theoretical optimal p (the '*' on the curve) is quite close to the numerical one in Figure 2.3.2.

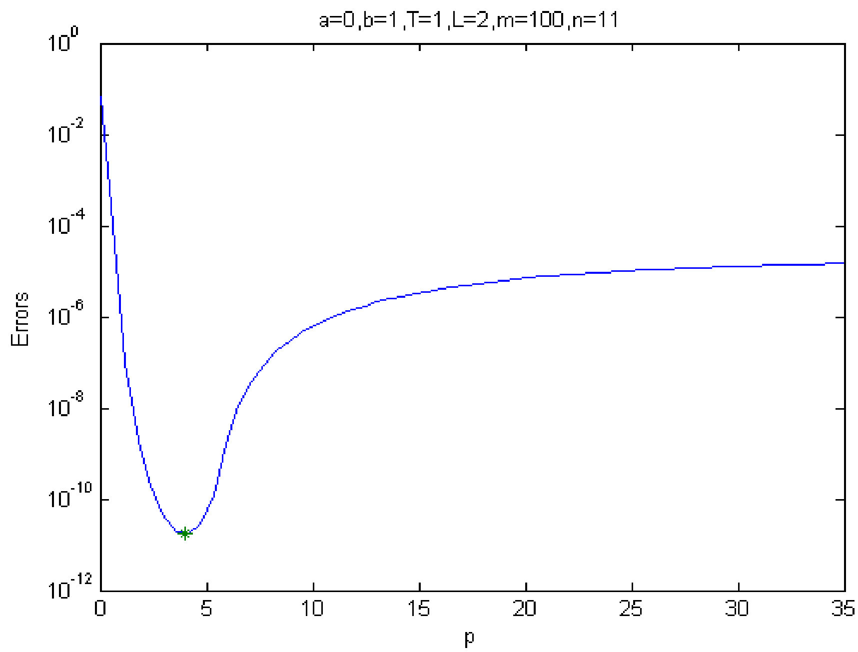

a=0,b=1,T=1,L=2,m=100,n=11
Errors
p
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
0
5
10
15
20
25
30
35

Figure 2.3.2

2.3.3 Test 3

Now, we again test the accuracy of our computation for the optimized Robin parameters, but for the nonoverlapping case. Since we would like also test the effect of different numbers of iterations on the optimized parameters, we plot the errors after 5, 8, 11, 14, 17, 20 interations. On the figures, we let p vary from 0 to 35, the theoretical optimal p is the star $*$ on the curve. The test corresponds to the case $dt = dx$. As we can see in Figure 2.3.3 that the optimized parameter does not depend on the number of iterations and this verifies our theoretical results.

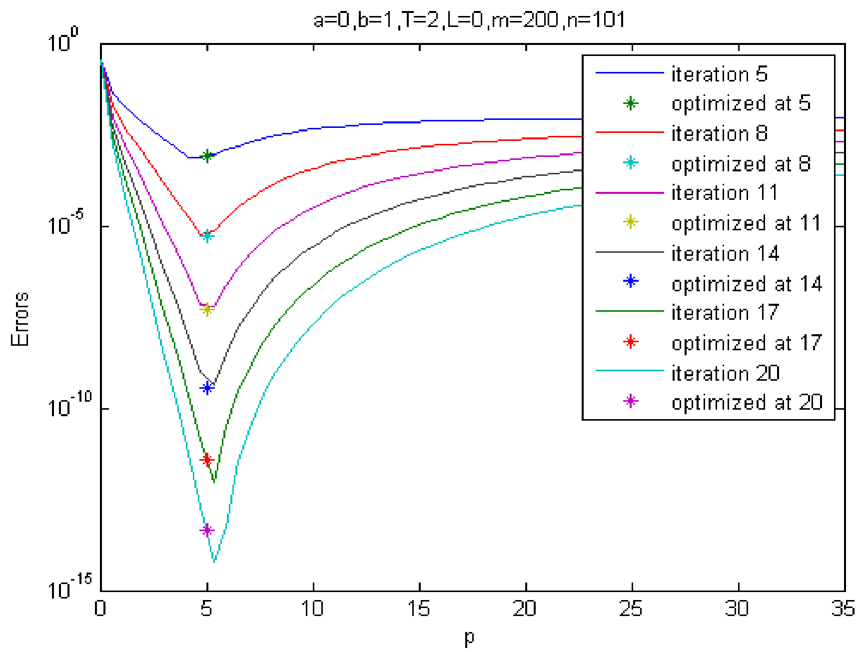

a=0,b=1,T=2,L=0,m=200,n=101
10^{0}
10^{-5}
10^{-10}
10^{-15}
Errors
0
5
10
15
20
25
30
35
p
iteration 5
optimized at 5
iteration 8
optimized at 8
iteration 11
optimized at 11
iteration 14
optimized at 14
iteration 17
optimized at 17
iteration 20
optimized at 20

Figure 2.3.3

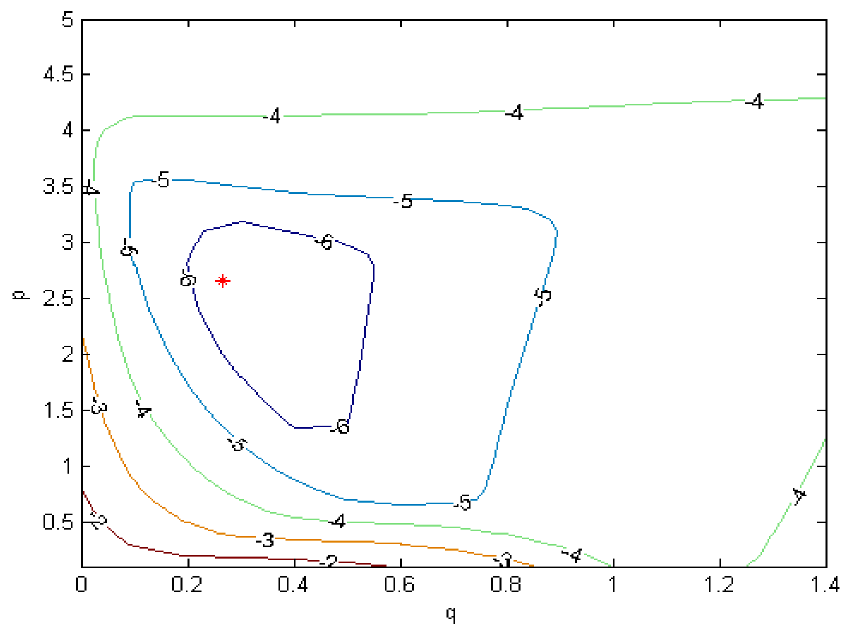
2.3.4 Test 4

In this test, we test our theoretical parameters for Ventcell transmission conditions. We choose the overlapping length to be 2 grid points. We plot the errors respect to p varying from 0 to 5 and q varying from 0 to 1.4 after 5 iterations. The the following two tests, we would like to test our theoretical results for both cases $dt = Cdx$ and $dt = Cdx^2$.

In the first case (Figure 2.3.4.A.), we choose 500 grid points on the time interval and 300 on the space interval, $dt = 0.01$, $dx = 0.01 = dt$.

In the second case (Figure 2.3.4.B), we choose 500 grid points on the time interval and 30 on the space interval, $dt = 0.1$, $dx = 0.01 = dt^2$.

We can see that in both cases the theoretical optimal (p, q) (the '*' on the curve) is quite close to the numerical one.



5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
p
0
0.2
0.4
0.6
0.8
1
1.2
1.4
q
-4
-4
-4
-4
-5
-5
-5
-5
-5
-5
-6
-6
-6
-4
-4
-4
-4
-3
-3
-3
-2
-2

Figure 2.3.4.A

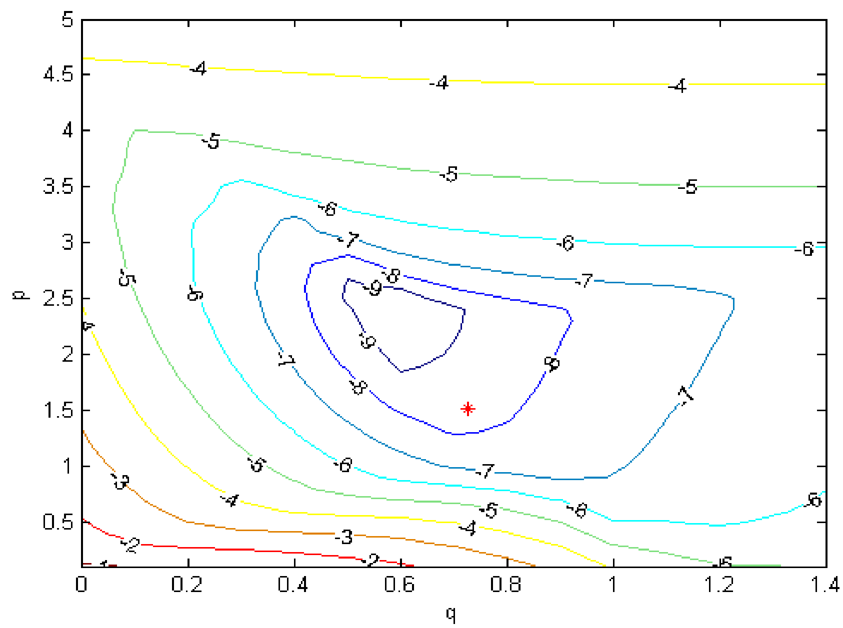


Figure 2.3.4.B

2.3.5 Test 5

Since, according to our theoretical results, the optimized parameters depend on the overlapping length, we consider our problems with different overlapping lengths in the tests.

In this test, we consider again the heat equation in 1D, $\nu = 1$, with Euler backward scheme, and Ventcell transmission conditions, for the domain $[0, 1]$, $T = 1$, 10 iterations. We choose 100 grid points on the time interval and 100 on the space interval. We plot the errors respect to p varying from 0 to 10 and q varying from 0 to 0.5.

In the first case (Figure 2.3.5.A.), we choose the overlapping length to be 4 grid points.

In the second case (Figure 2.3.5.B.), we choose the overlapping length to be 3 grid points.

In the third case (Figure 2.3.5.C.), we choose the overlapping length to be 2 grid points.

In the forth case (Figure 2.3.5.D.), we choose the overlapping length to be 1 grid points. We can see that the theoretical optimal (p, q) (the stars in the pictures) work quite well.

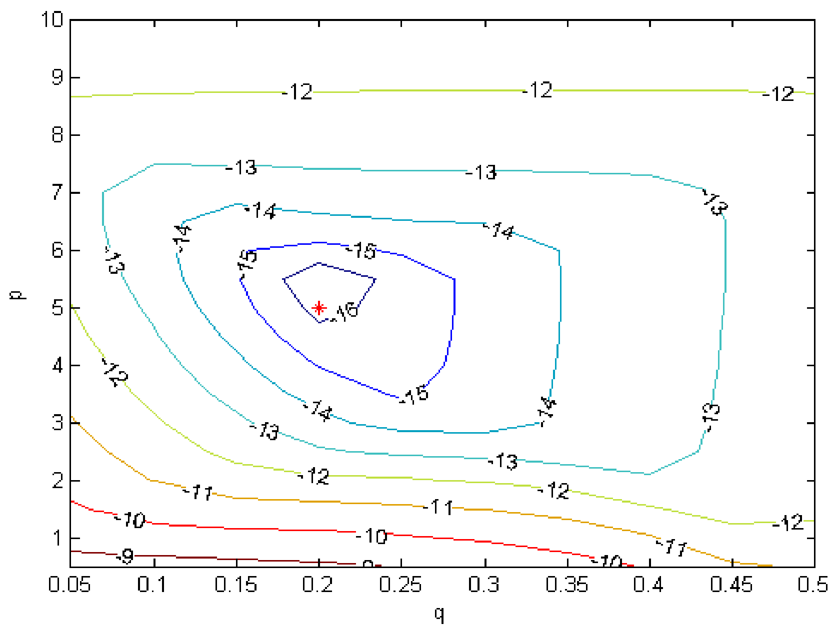


Figure 2.3.5.A

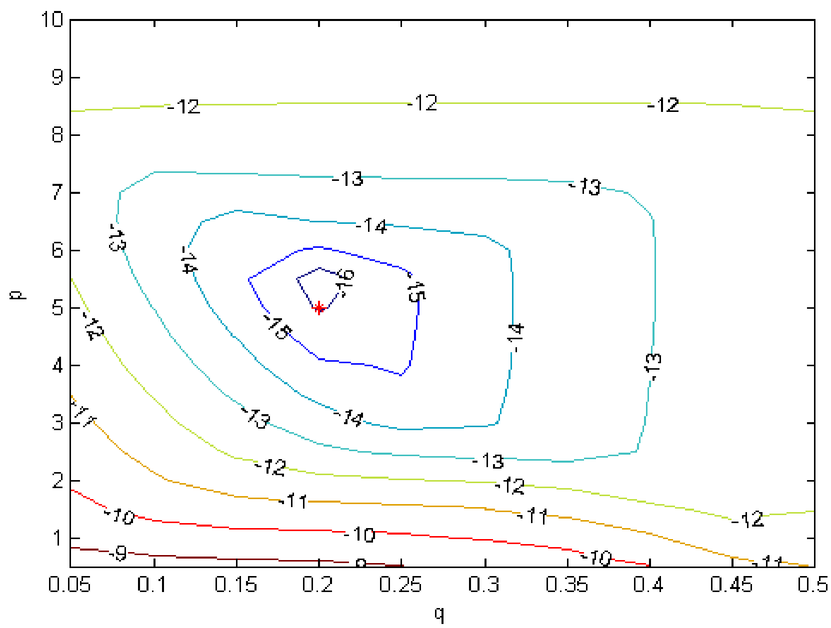


Figure 2.3.5.B

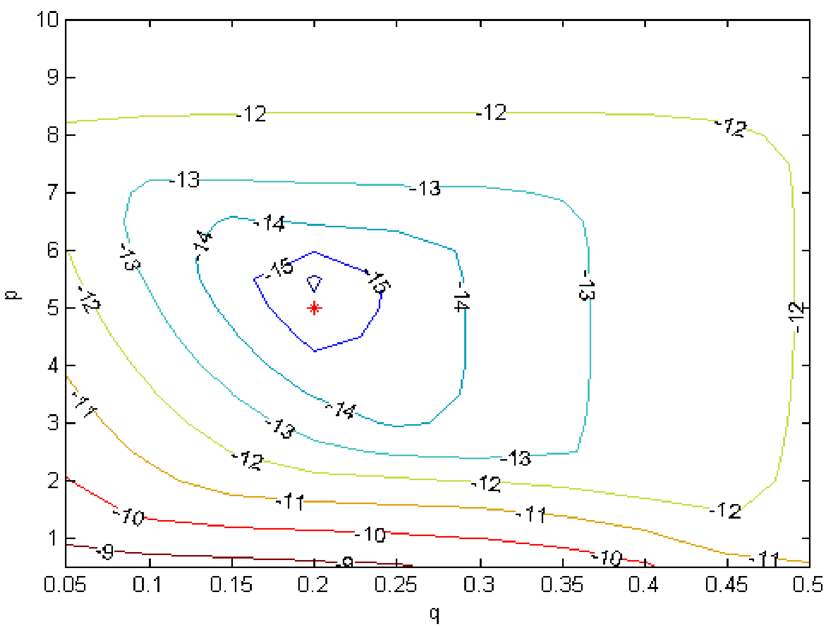
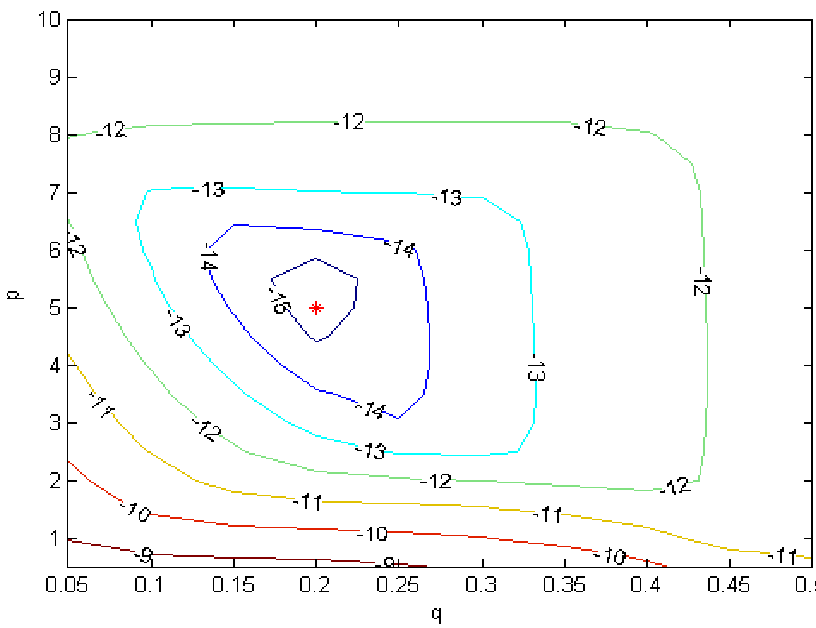

10
9
8
7
6
5
4
3
2
1
p
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
q
-12
-13
-14
-15
-11
-10
-9

Figure 2.3.5.C



10
9
8
7
6
5
4
3
2
1
p
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
q
-12
-12
-12
-12
-12
-12
-12
-13
-13
-13
-13
-13
-14
-14
-14
-15
-11
-11
-11
-11
-10
-10
-10
-9
-9

Figure 2.3.5.D

2.3.6 Test 6

According to our theoretical results, the optimized parameters have the asymptotic behavior of $Cdx^{-/13}$ and $Cdx^{-1/4}$. In this test, we want to verify this.

We consider 100 grid points in the space interval and 200 grid points in the time interval, then $dx = dt = 0.01$ and fixed the overlapping length to be 2 grid points. We repeat this experiment by dividing dx and dt by 2, 3, 4, 5. We plot the practical optimized parameters according to each mesh size and the line $p = dx^{-1/4}$. We can see on Figure 2.3.6A that the practical optimized line and the line $p = dx^{-1/4}$ are parallel. Which means that the asymptotic analysis predicts very well the behavior of the optimized algorithm.

We consider the same experiment but with 10 grid points in the space interval and 200 on the time interval, then $dt = dx^2 = 0.01$, the overlapping length is again 2 grid points. We repeat this experiment by dividing dx and dt by 2, 3, 4, 5. We plot the practical optimized parameters according to each mesh size and the line $p = dx^{-1/3}$. The asymptotic analysis again predicts very well the behavior of the optimized algorithm in this case.

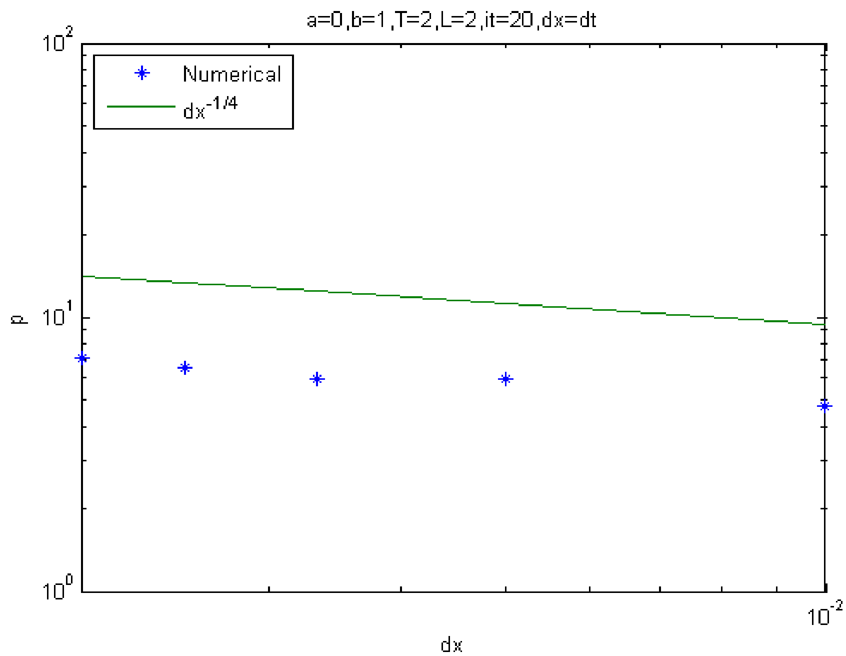
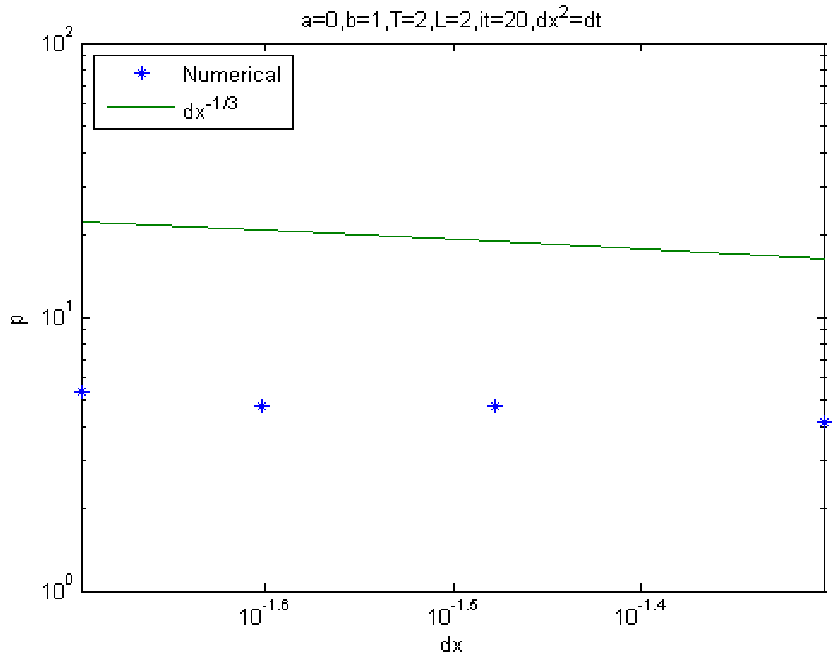


Figure 2.3.6.A



a=0,b=1,T=2,L=2,it=20,dx^2=dt
Numerical
dx^-1/3
p
10^2
10^1
10^0
10^-1.6
10^-1.5
10^-1.4
dx

Figure 2.3.6.B

2.3.7 Test 7

As predicted in our theoretical results, the performance of the optimized Schwarz methods depend on the lengths of the time intervals, we now do some tests on this. We will increase the length of the time intervals, but keep the same dt, and look at the behavior of the methods at each case.

In 2.3.7.A, we consider 10 grid points in the space interval and 100 grid points in the time interval, then $dx^2 = dt = 0.01$ and fixed the overlapping length to be 2 grid points. Then we plot the errors of the methods with respect to the number of iteration. We increase the time interval from $[0, 1]$ to $[0, 10]$ and choose 1000 grid points on the time interval and plot the second curve. We increase the time interval from $[0, 1]$ to $[0, 20]$ and choose 2000 grid points on the time interval. We can see that the behavior of the methods depend on the length of the time interval and plot the third curve. We can see that the behavior of the methods depend on the length of the time interval.

In 2.3.7.B, we increase the time interval from $[0, 1]$ to $[0, 16]$ and choose 1600 grid points on the time interval and plot the errors of the methods with respect to the Robin parameters. We plot 3 curves with respect to the 5, 8, 11 iterations. We can see that our theoretical p (the '*' in the picture) predicts well the practical optimized parameter.

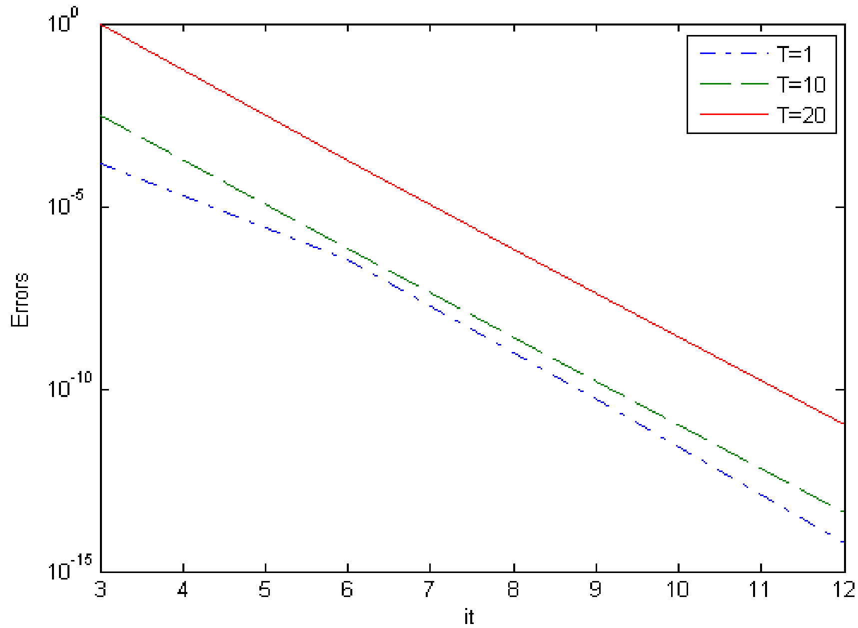


Figure 2.3.7.A

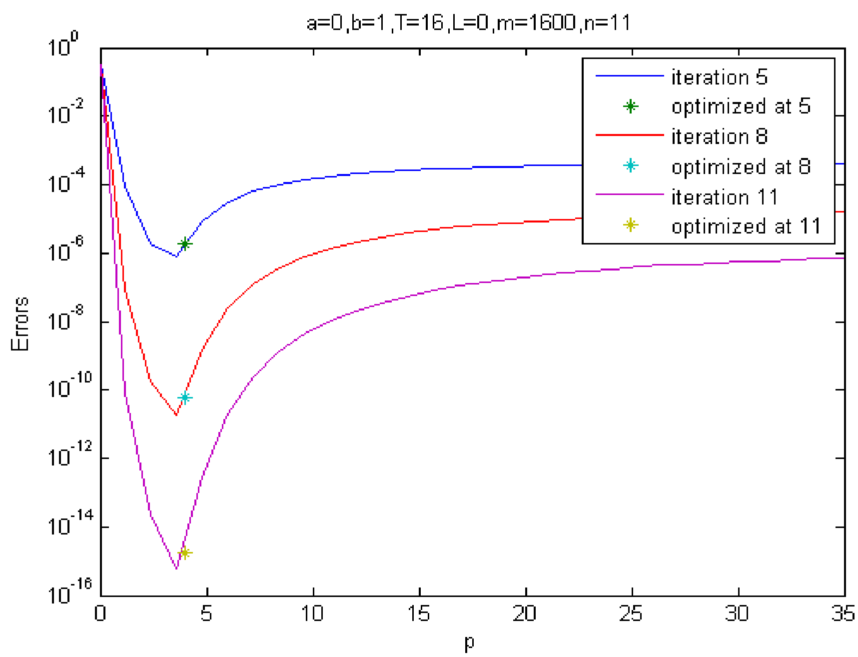

a=0,b=1,T=16,L=0,m=1600,n=11
iteration 5
optimized at 5
iteration 8
optimized at 8
iteration 11
optimized at 11
Errors
p

Figure 2.3.7.B

2.3.8 Test 8

As predicted in our theoretical results, the performance of the optimized Schwarz methods depend on the parameter ν also, we now do some tests on this.

In picture 2.3.8.A, we consider 10 grid points in the space interval and 10 grid points in the time interval, then $dx = dt = 0.1$ and fixed the overlapping length to be 2 grid points. Then we plot the errors of the methods with respect to the number of iteration for three cases $\nu = 0.1$, $\nu = 1$, $\nu = 10$. We can see that the performance of the algorithm really depends on the viscosity parameter.

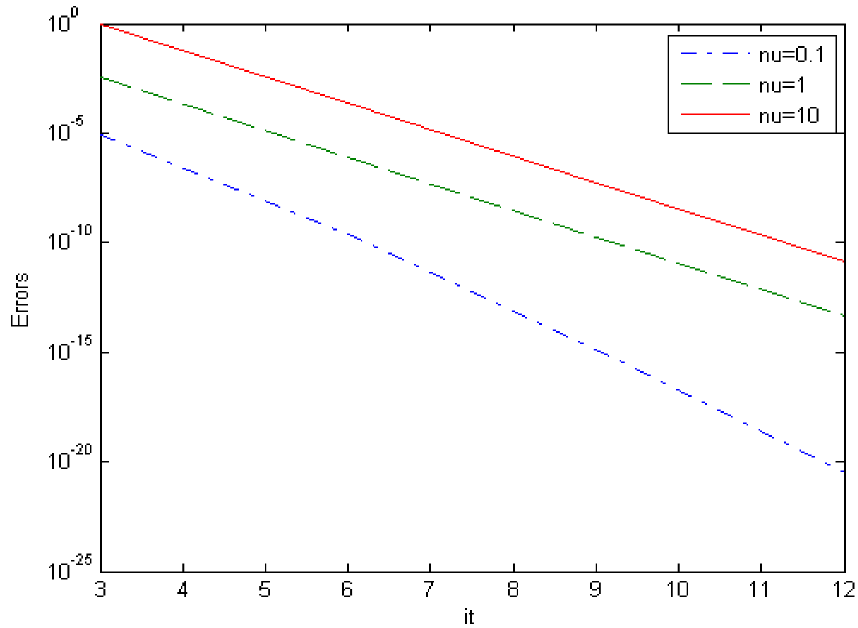


Figure 2.3.8.A

2.4 Optimization of The Convergence Factor: A Theoretical Attempt

2.4.1 The results

We define the convergence factor by

$$\rho(\omega, p, L) = |(\frac{2\sqrt{i\omega\nu} - p}{2\sqrt{i\omega\nu} + p})^2 \exp(-2L\sqrt{\frac{i\omega}{\nu}})| \tag{2.4.1}$$

$$= \exp(-L\sqrt{\frac{2|\omega|}{\nu}})\frac{(\sqrt{2|\omega|\nu} - p)^2 + 2|\omega|\nu}{(\sqrt{2|\omega|\nu} + p)^2 + 2|\omega|\nu}.$$

We have

$$\rho(\omega, p, L) = \exp(-L\sqrt{\frac{2|\omega|}{\nu}})\frac{(L\sqrt{\frac{2|\omega|}{\nu}} - \frac{p}{\nu}L)^2 + L^2\frac{2|\omega|}{\nu}}{(L\sqrt{\frac{2|\omega|}{\nu}} + \frac{p}{\nu}L)^2 + L^2\frac{2|\omega|}{\nu}}. \tag{2.4.2}$$

And we have to solve the following min-max problem

$$\min_p \max_{|\omega|\in[\frac{\pi}{T}, \frac{\pi}{\triangle t}]} \{ \exp(-L\sqrt{\frac{2\omega}{\nu}})\frac{(L\sqrt{\frac{2|\omega|}{\nu}} - \frac{p}{\nu}L)^2 + L^2\frac{2|\omega|}{\nu}}{(L\sqrt{\frac{2|\omega|}{\nu}} + \frac{p}{\nu}L)^2 + L^2\frac{2|\omega|}{\nu}}\}. \tag{2.4.3}$$

Without loss of generality, we can assume that $\omega \geq 0$
Put $x = L\sqrt{\frac{2\omega}{\nu}}$. Since $\omega \in [\frac{\pi}{T}, \frac{\pi}{\triangle t}]$, we have $x \in [L\sqrt{\frac{2\pi}{T\nu}}, L\sqrt{\frac{2\pi}{\triangle t\nu}}]$.
Then

$$\rho(\omega, p, L) = \exp(-x)\frac{(x - a)^2 + x^2}{(x + a)^2 + x^2} \tag{2.4.4}$$

$$= \exp(-x)\frac{2x^2 - 2xa + a^2}{2x^2 + 2xa + a^2}.$$

We define

$$\alpha_0 = \sqrt{3} - 1, \qquad a = \frac{pL}{\nu},$$

$$\alpha = L\sqrt{\frac{2\pi}{T\nu}}, \qquad \beta = L\sqrt{\frac{2\pi}{\Delta t\nu}}, \qquad k = \frac{\beta}{\alpha}.$$

For $a \in \mathbb{R}$, $x \in \mathbb{R}$, we define:

$$R(a,x) = exp(-x)\frac{2x^2 - 2xa + a^2}{2x^2 + 2xa + a^2}$$

Instead of solving the problem (2.4.3), we will solve the following min-max problem:

$$\min_a \max_{x \in [L\sqrt{\frac{2\pi}{T\nu}}, L\sqrt{\frac{2\pi}{\triangle t\nu}}]} R(a,x). \tag{2.4.5}$$

Suppose that $\alpha \leq 1$, $\beta \leq 1$ we define

$$h_1 = -1 + \sqrt{1-\alpha^2} + (2 + \alpha^2 - 2\sqrt{1-\alpha^2})^{\frac{1}{2}},$$

$$h_2 = -1 - \sqrt{1-\alpha^2} + (2 + \alpha^2 + 2\sqrt{1-\alpha^2})^{\frac{1}{2}},$$

$$k_1 = -1 + \sqrt{1-\beta^2} + (2 + \beta^2 - 2\sqrt{1-\beta^2})^{\frac{1}{2}},$$

$$k_2 = -1 - \sqrt{1-\beta^2} + (2 + \beta^2 + 2\sqrt{1-\beta^2})^{\frac{1}{2}}.$$

Suppose that $0 \leq a \leq 2\sqrt{2} - 2$, we define

$$X_1(a) = \frac{2a - a\sqrt{4 - 4a - a^2}}{2},$$

$$X_2(a) = \frac{2a + a\sqrt{4 - 4a - a^2}}{2},$$

and $x_1 = \sqrt{X_1}$ and $x_2 = \sqrt{X_2}$.

We define B_1 and B_2 to be the solutions of $R(k_1, x_2) = R(k_1, B_1)$ and $R(k_2, x_2) = R(k_2, B_2)$ (see Lemma 2.4.16).

Theorem 2.4.1. *We denote S_1 to be the solution of*

$$R(S_1, \alpha) = R(S_1, \beta) \text{ in } [\sqrt{2}\alpha, \sqrt{2}\beta].$$

And S_2 is the solution of

$$R(S_2, \alpha) = R(S_2, x_2) \text{ in } [\sqrt{2}\alpha, \infty),$$

(see Lemma 2.4.17*).*

The problem (2.4.5) *has a unique solution which depends on the following* *cases*

Case 1: $\alpha < \beta \leq 1$ *we have the following disjoint cases*

Case 1.1 If $\alpha < \min\{\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})), \frac{2\sqrt{2}(k^2-1)}{k^4+1}\}$, $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

Case 1.2 If $\alpha < \min\{\frac{k_2}{\sqrt{2}}, B_2, M_0, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_2,\alpha)$.

Case 1.3 If $B_2 \leq \alpha < \min\{\frac{k_2}{\sqrt{2}}, M_0, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

Case 1.4 Otherwise $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

Case 2: $\alpha \leq 1 < \beta$

Case 2.1 If $\alpha \leq M_0$ $\min_a \max_x R(a,x) = R(S_2,\alpha)$.

Case 2.2 If $\alpha > M_0$ $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

Case 3: $1 < \alpha < \beta$ $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

Remark 2.4.1. *Actually, for* $\alpha < \beta \leq 1$, *we have the following cases with their figures*

* *If* $\alpha < \min\{\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})), \frac{2\sqrt{2}(k^2-1)}{k^4+1}\}$, $\min_a \max_x R(a,x) = R(S_1,\alpha)$ *(see figure* 1*).*

* *If* $\beta \leq \sqrt{2\sqrt{2}-2}$ *and* $\alpha < B_2$ $\min_a \max_x R(a,x) = R(S_2,\alpha)$ *(see figure* 2*)* *and if* $B_2 \leq \alpha < \min\{\frac{k_2}{\sqrt{2}}, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_1,\alpha)$ *(see figure* 3*).*

* *If* $\beta > \sqrt{2\sqrt{2}-2}$ *and* $\alpha < \min\{\frac{k_2}{\sqrt{2}}, B_2, M_0, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_2,\alpha)$ *(see figure* 4*) and if* $\beta > \sqrt{2\sqrt{2}-2}$ *and* $B_2 \leq \alpha < \min\{\frac{k_2}{\sqrt{2}}, M_0, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_1,\alpha)$ *(see figure* 4*).*

* *Otherwise* $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

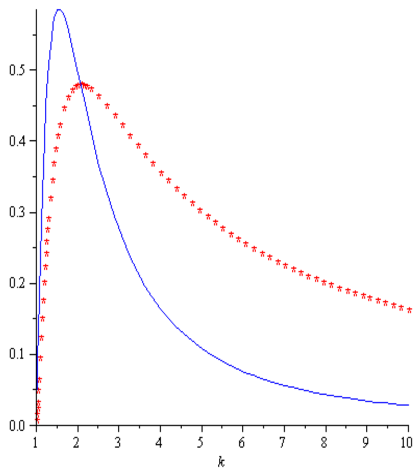


Figure 2.4.1. The domain in the first case: $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2}))$ *(asterisk),* $\frac{2\sqrt{2}(k^2-1)}{k^4+1}$ *(line),* $\beta \in [0, \sqrt{2\sqrt{2}-2}]$*.*

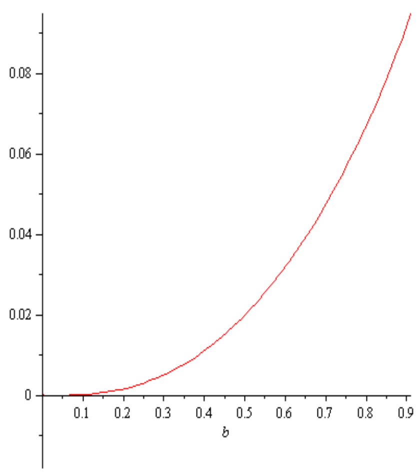


Figure 2.4.2. The domain in the second case: the function in the figure is B_2*,* $\beta \in [0, \sqrt{2\sqrt{2}-2}]$*.*

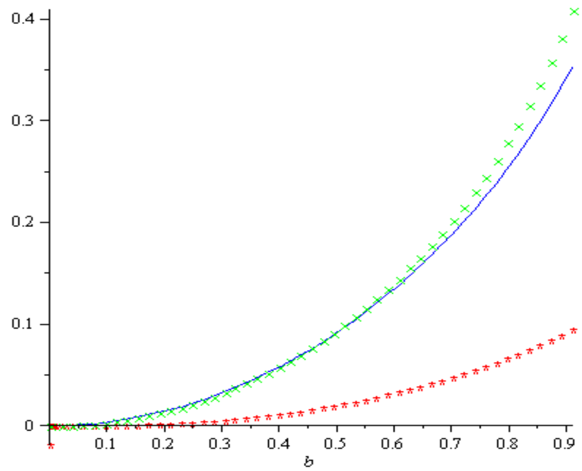


Figure 2.4.3. The domain in the third case: B_2 *(asterisk),* $\frac{k_2}{\sqrt{2}}$ *(line) and* $\sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2\}}$ *(cross),* $\beta \in [0, \sqrt{2\sqrt{2}-2}]$.

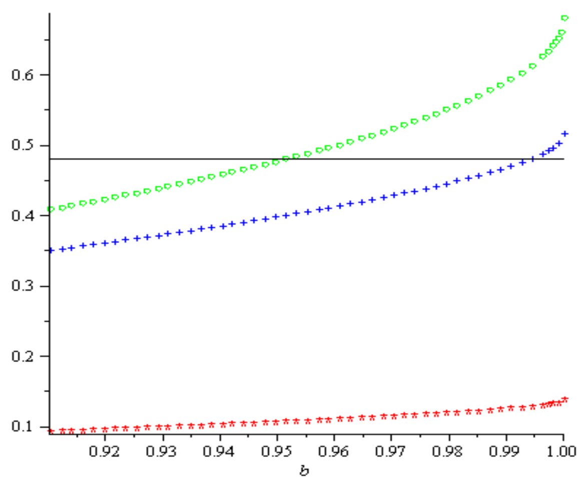


Figure 2.4.4. The domain in the forth and fifth cases: B_2 *(asterisk),* $\frac{k_2}{\sqrt{2}}$ *(cross) and* $\sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}$ *(circle),* $\beta \in [\sqrt{2\sqrt{2}-2}, 1]$.

2.4.2 Proofs of the results

We need the following Lemmas:

Lemma 2.4.1. *If $\alpha \leq 1$ we have that $h_1 \geq h_2$. And if $\beta \leq 1$ we have that $k_1 \geq k_2$*

Proof.
We prove that $h_1 > h_2$ ($k_1 \geq k_2$ can be proven similarly.)
We have

$$2 - 2\sqrt{1-\alpha^2} + 2\sqrt{1-\alpha^2}(2+\alpha^2 - 2\sqrt{1-\alpha^2})^{\frac{1}{2}} > 0.$$

This can be written as

$$(\sqrt{1-\alpha^2} + (2+\alpha^2 - 2\sqrt{1-\alpha^2})^{\frac{1}{2}})^2 > 1.$$

Take the square root on both sides of this and multify the result by $\sqrt{1-\alpha^2}$, we have

$$1 - \alpha^2 - \sqrt{1-\alpha^2} + \sqrt{1-\alpha^2}(2+\alpha^2 - 2\sqrt{1-\alpha^2})^{\frac{1}{2}} > 0.$$

This can be developed into

$$(2\sqrt{1-\alpha^2} + (2+\alpha^2 - 2\sqrt{1-\alpha^2})^{\frac{1}{2}})^2 > 2 + \alpha^2 + 2\sqrt{1-\alpha^2}.$$

Take the square root on both sides of this, we have that

$$-1 + \sqrt{1-\alpha^2} + (2+\alpha^2 - 2\sqrt{1-\alpha^2})^{\frac{1}{2}} > -1 - \sqrt{1-\alpha^2} + (2+\alpha^2 + 2\sqrt{1-\alpha^2})^{\frac{1}{2}}.$$

■

Lemma 2.4.2. *For $y \geq 0$, we put $F_y(x) = x^4 + 4x^3 - 8yx + 4y^2$. If $0 \leq y < 1$ then the equation $F_y(x) = 0$ has two positive solutions. If $0 \leq y < 1$, then $F_y(x) > 0$ for all $x \geq 0$. If $y = 1$, the equation has one positive solution $x = \sqrt{3} - 1 =: \alpha_0$.*

Proof. Put $\alpha_0 = \sqrt{3} - 1$, we have $\alpha_0^3 + 6\alpha_0^2 + 6\alpha_0 - 8 = 0$.

Let y_1 be the unique positive solution of $y = \frac{y_1^3 + 3y_1^2}{2}$. We have $F'y(x) = 4x^3 + 12x^2 - 8y = 4(x^3 + 3x^2 - 2y)$, and we have the following table

a		y_1	
F'_y	-	0	+
F_y	$\searrow$		$\nearrow$
		$F_y(y_1)$	

Case 1: If $0 \leq y < 1$, then $y < \frac{\alpha_0^3 + 3\alpha_0^2}{2}$. We can see that $y_1 < \alpha_0$.
We have that

$$F_y(y_1) = y_1^4 + 4y_1^3 - 8y_1 y + 4y^2 = y_1^3(y_1^3 + 6y_1^2 + 6y_1 - 8)$$

Since $y_1 < \alpha_0$, we have $F_y(y_1) < 0$.
Thus, the equation has two positive solutions.
Case 2: If $y > 1$, then $y > \frac{\alpha_0^3 + 3\alpha_0^2}{2}$. We can see that $y_1 > \alpha_0$.
We have that

$$F_y(y_1) = y_1^4 + 4y_1^3 - 8y_1 y + 4y^2 = y_1^3(y_1^3 + 6y_1^2 + 6y_1 - 8)$$

Since $y_1 > \alpha_0$, we have $F_y(y_1) > 0$.
Thus, the equation has no positive solution.
Case 3: If $y = 1$, the equation has one postive solution $y_1 = \alpha_0 = \sqrt{3} - 1$.
■

Lemma 2.4.3. *Suppose that x is in $\mathbb{R}_+$ and $f(x) = \frac{2 - 2x + x^2}{2 + 2x + x^2}$. Then f decreases in $[0, \sqrt{2}]$ and increases in $[\sqrt{2}, \infty)$.*

Proof. We have

$$f'(x) = \frac{4(x^2 - 2)}{(x^2 + 2x + 2)^2},$$

which means that $f' \leq 0$ if $x \in [0, \sqrt{2}]$ and $f' \geq 0$ if $x \in [\sqrt{2}, \infty)$.
■

Lemma 2.4.4. *Let $M_0 = 0.481033790$ be the solution of $R(\sqrt{2}M_0, x_2(\sqrt{2}M_0)) = R(\sqrt{2}M_0, M_0)$.*
If $\alpha < M_0$, then $R(\sqrt{2}\alpha, x_2(\sqrt{2}\alpha)) > R(\sqrt{2}\alpha, \alpha)$.
If $1 \geq \alpha \geq M_0$, then $R(\sqrt{2}\alpha, x_2(\sqrt{2}\alpha)) \leq R(\sqrt{2}\alpha, \alpha)$.

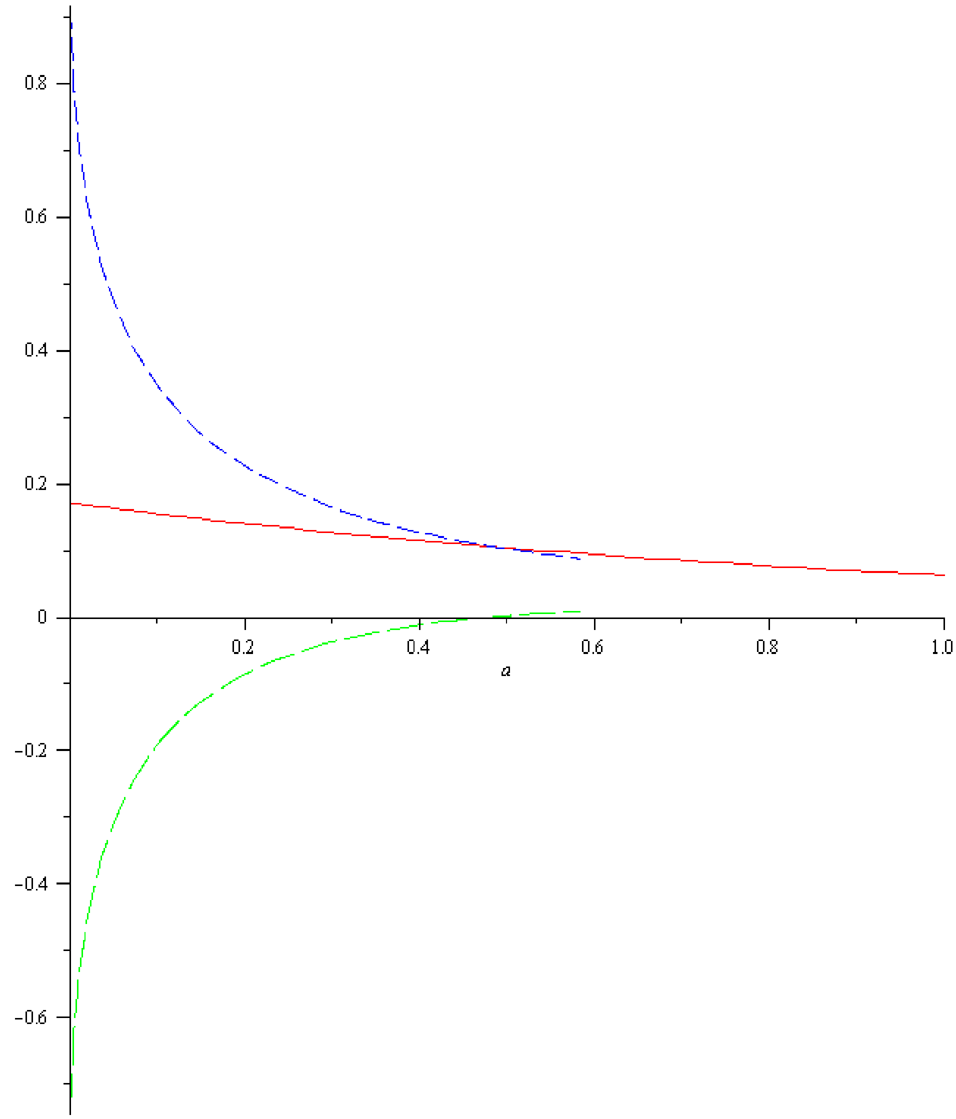


Figure 2.4.5. The graph of $R(\sqrt{2}\alpha, x_2(\alpha))$ *(dash),* $R(\sqrt{2}\alpha, \alpha)$ *(line), and the substraction (long dash).*

Lemma 2.4.5. *$k_2 \leq \sqrt{2}\alpha \leq k_1$ iff $\alpha \leq \frac{2\sqrt{2}(k^2-1)}{k^4+1}$.*

Proof. We have that $k_2 \leq \sqrt{2}\alpha \leq k_1$ is equivalent to

$$(\sqrt{2}\alpha)^4 + 4(\sqrt{2}\alpha)^3 - 8\beta^2\sqrt{2}\alpha + 4\beta^4 \leq 0,$$

or

$$(\sqrt{2}\alpha)^4 + 4(\sqrt{2}\alpha)^3 - 8k^2\sqrt{2}\alpha^3 + 4k^4\alpha^4 \leq 0.$$

This means

$$4\alpha + 8\sqrt{2} - 8\sqrt{2}k^2 + 4k^4\alpha \leq 0,$$

or

$$\alpha \leq \frac{2\sqrt{2}(k^2-1)}{k^4+1}.$$

■

Lemma 2.4.6. *If $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})) \leq \alpha$, we have $R(\sqrt{2}\alpha,\beta) \leq R(\sqrt{2}\alpha,\alpha)$.*

If $\alpha < \frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2}))$, we have $R(\sqrt{2}\alpha,\beta) > R(\sqrt{2}\alpha,\alpha)$.

Proof. We have that

$$R(\sqrt{2}\alpha,\beta) = \exp(-\beta)\frac{2\beta^2 - 2\beta\sqrt{2}\alpha + 2\alpha^2}{2\beta^2 + 2\beta\sqrt{2}\alpha + 2\alpha^2}$$

$$= \exp(-k\alpha)\frac{k^2 - \sqrt{2}k + 1}{k^2 + \sqrt{2}k + 1}.$$

Moreover

$$R(\sqrt{2}\alpha,\alpha) = \exp(-\alpha)(3 - 2\sqrt{2}).$$

Thus

$$R(\sqrt{2}\alpha,\beta) \leq R(\sqrt{2}\alpha,\alpha)$$

iff

$$(3+2\sqrt{2})\frac{k^2 - \sqrt{2}k + 1}{k^2 + \sqrt{2}k + 1} \leq \exp((k-1)\alpha).$$

This is equivalent to

$$\frac{1}{k-1}\ln((3+2\sqrt{2})\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}) \leq \alpha.$$

Thus we get the result.

∎

Lemma 2.4.7. *For $k > 1$, we always have*

$$\frac{2\sqrt{2}(k^2-1)}{k^4+1} \leq \frac{1}{k},$$

$$\frac{2\sqrt{2}(k^2-1)}{k^4+1} < \sqrt{2\sqrt{2}-2},$$

and

$$\frac{1}{k-1}\ln((3+2\sqrt{2})\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}) \leq \frac{2\sqrt{2}(k^2-1)}{k^4+1} \text{ iff } k < M_1 = 2.065883380.$$

Proof. Since

$$0 \leq (k^2-\sqrt{2}k-1)^2,$$

we have

$$0 \leq k^4 - 2\sqrt{2}k^3 + 2\sqrt{2}k + 1.$$

This implies

$$\frac{2\sqrt{2}(k^2-1)}{k^4+1} \leq \frac{1}{k}.$$

We have that

$$\sqrt{\sqrt{2\sqrt{2}-2}}\sqrt{\sqrt{2\sqrt{2}-2}+2\sqrt{2}} > \sqrt{2}.$$

Thus

$$\sqrt{2\sqrt{2}-2}k^4+(\sqrt{2\sqrt{2}-2}+2\sqrt{2}) \geq 2\sqrt{\sqrt{2\sqrt{2}-2}}\sqrt{\sqrt{2\sqrt{2}-2}+2\sqrt{2}}k^2 > 2\sqrt{2}k^2.$$

Hence

$$\sqrt{2\sqrt{2}-2}(k^4+1) > 2\sqrt{2}(k^2-1).$$

The rest of the proof can be easily seen on Figure 5.6.

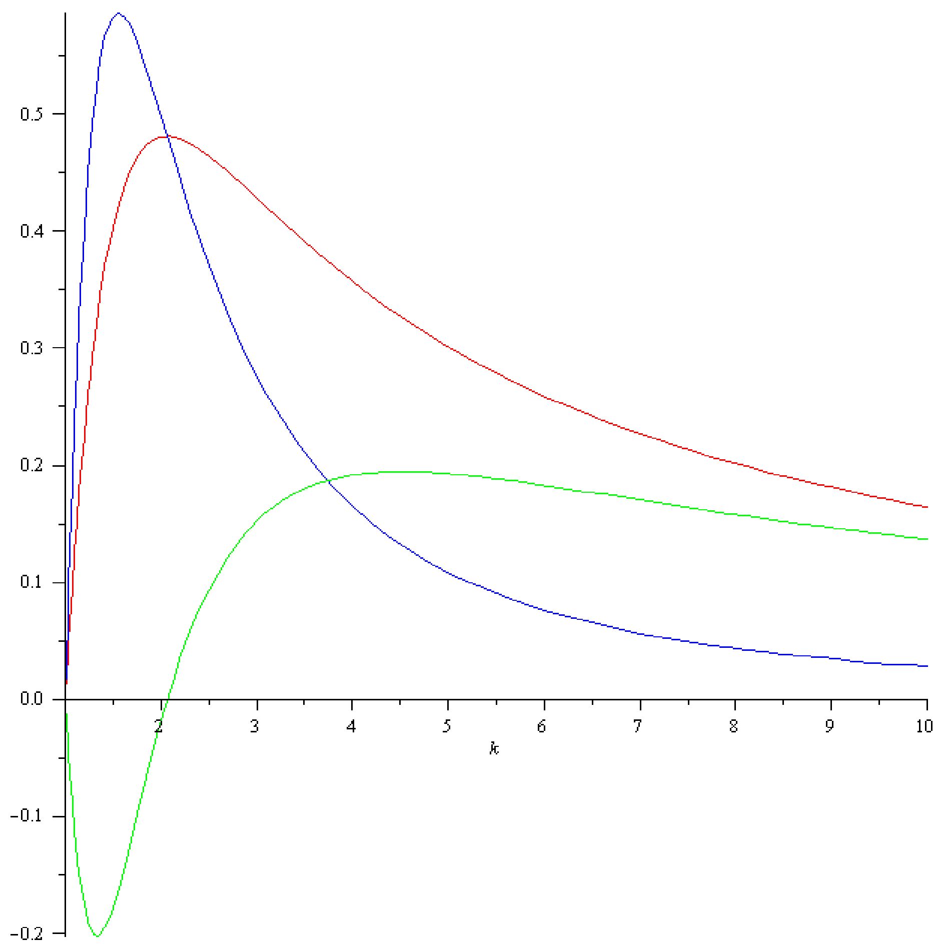


Figure 2.4.6. The two functions $\frac{1}{k-1}\ln((3+2\sqrt{2})\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1})$ and $\frac{2\sqrt{2}(k^2-1)}{k^4+1}$. ■

Lemma 2.4.8. *For $0 < \beta \leq 1$, we have that $\beta^2 > k_2$.*

For $\sqrt{2\sqrt{2}-2} \leq \beta \leq 1$, we have that $\beta^2 \geq k_1$; and for $0 < \beta < \sqrt{2\sqrt{2}-2}$, we have that $\beta^2 < k_1$.

Proof.

*For $0 < \beta \leq 1$, we have that

$$0 < 2\beta^2\sqrt{1-\beta^2} + \beta^4.$$

Thus

$$2 + \beta^2 + 2\sqrt{1-\beta^2} < 1 + \beta^4 + 1 - \beta^2 + 2\beta^2 + 2\sqrt{1-\beta^2} + 2\beta^2\sqrt{1-\beta^2}.$$

Which implies

$$\sqrt{2 + \beta^2 + 2\sqrt{1-\beta^2}} < 1 + \beta^2 + \sqrt{1-\beta^2}.$$

This means $\beta^2 > k_2$.

*For $1 \geq \beta \geq \sqrt{2\sqrt{2}-2}$, we have

$$\beta^4 + 4\beta^2 - 4 \geq 0,$$

or

$$\beta^2 \geq 2\sqrt{1-\beta^2}.$$

Thus

$$\beta^4 + 1 + 1 - \beta^2 + 2\beta^2 - 2\sqrt{1-\beta^2} - 2\beta^2\sqrt{1-\beta^2} \geq 2 + \beta^2 - 2\sqrt{1-\beta^2},$$

or

$$(\beta^2 + 1 - \sqrt{1-\beta^2})^2 \geq 2 + \beta^2 - 2\sqrt{1-\beta^2}.$$

Hence $\beta^2 \geq k_1$.

*For $0 < \beta < \sqrt{2\sqrt{2}-2}$, by a similar way, we can prove that $\beta^2 < k_1$. ∎

Lemma 2.4.9. *$h_1 \leq 2\sqrt{2}-2$ and $k_1 \leq 2\sqrt{2}-2$.*

Proof.
We prove for the case $h_1 \leq 2\sqrt{2} - 2$, the other case is similar.
This is equivalent to

$$-1 + \sqrt{1 - \alpha^2} + (2 + \alpha^2 - 2\sqrt{1 - \alpha^2})^{\frac{1}{2}} \leq 2\sqrt{2} - 2.$$

We develop this

$$\frac{1 + \sqrt{1 - \alpha^2}}{2} + (1 - (\frac{1 + \sqrt{1 - \alpha^2}}{2})^2)^{\frac{1}{2}} \leq \sqrt{2}.$$

We put $y = \frac{1+\sqrt{1-\alpha^2}}{2}$, $y \in [0, 1]$. Then this can be rewritten into

$$y + (1 - y^2)^{\frac{1}{2}} \leq \sqrt{2}.$$

This is equivalent to

$$1 + 2y(1 - y^2)^{\frac{1}{2}} \leq 2,$$

which is obviously true.

■

Lemma 2.4.10. *We have that $h_1 < \sqrt{2\alpha^2}$ and $k_1 < \sqrt{2\beta^2}$.*

Proof
We prove for the case $h_1 < \sqrt{2\alpha^2}$, the other case is similar.
Put $x = \sqrt{1 - \alpha^2}$, we have to prove that for all x in $[0, 1]$

$$-1 + x + (3 - 2x - x^2)^{\frac{1}{2}} < \sqrt{2(1 - x^2)},$$

or

$$-1 + x + ((3 + x)(1 - x))^{\frac{1}{2}} < \sqrt{2(1 - x^2)},$$

or

$$\sqrt{3 + x} - \sqrt{1 - x} < \sqrt{2(1 + x)},$$

or

$$3 + x + 1 - x - 2\sqrt{3 + x}\sqrt{1 - x} < 2(1 + x),$$

or

$$(1 - x)^{\frac{1}{2}} < (3 + x)^{\frac{1}{2}}.$$

The last one is obviuous.

■

Lemma 2.4.11. *$f(\alpha) = R(\sqrt{2\sqrt{2}-2}, \alpha)$ is a decreasing function. If $\alpha < \sqrt{2\sqrt{2}-2}$ we have $R(2\sqrt{2}-2, \alpha) > R(2\sqrt{2}-2, x_2)$, and if $\alpha \geq \sqrt{2\sqrt{2}-2}$ we have $R(2\sqrt{2}-2, \alpha) \leq R(2\sqrt{2}-2, x_2)$.*

Proof. We have that

$$x_2 = \sqrt{2\sqrt{2}-2 + \frac{(2\sqrt{2}-2)\sqrt{4-4(2\sqrt{2}-2)-(2\sqrt{2}-2)^2}}{2}} = \sqrt{2\sqrt{2}-2}.$$

We have

$$f(\alpha) = R(2\sqrt{2}-2, \alpha) = -\exp(-\alpha)\frac{2\alpha^2 - 2\alpha(2\sqrt{2}-2) + (2\sqrt{2}-2)^2}{2\alpha^2 + 2\alpha(2\sqrt{2}-2) + (2\sqrt{2}-2)^2}.$$

Therefore

$$f'(\alpha) = -\exp(-\alpha)\frac{4(\alpha^2 - (2\sqrt{2}-2))^2}{(2\alpha^2 + 2\alpha^2\sqrt{2\sqrt{2}-2} + 2\sqrt{2}-2)^2}.$$

From this we get the result.

■

Lemma 2.4.12. *For $\alpha < \beta \leq 1$ and $0 \leq a \leq 2\sqrt{2}-2$, we have:*

$\alpha < \beta \leq x_1 \leq x_2$ iff $\beta \leq \sqrt{2\sqrt{2}-2}$ and $k_1 \leq a \leq 2\sqrt{2}-2$.

$x_1 \leq x_2 \leq \alpha < \beta$ iff $a \leq h_2$ and $h_1 \leq a \leq 2\sqrt{2}-2$ for $\sqrt{2\sqrt{2}-2} \leq \alpha$; and $a \leq h_2$ for $\alpha < \sqrt{2\sqrt{2}-2}$.

Proof.

The condition of $\alpha < \beta \leq x_1 \leq x_2$.

$\alpha < \beta \leq x_1 \leq x_2$ is equivalent to

$$\beta^2 \leq a - \frac{a\sqrt{4-4a-a^2}}{2},$$

or

$$\begin{cases} a \geq \beta^2, \\ a^4 + 4a^3 - 8a\beta^2 + 4\beta^4 \geq 0. \end{cases}$$

This is equivalent to

$$\begin{cases} a \geq \beta^2, \\ a \geq k_1 \text{ or } a \leq k_2. \end{cases}$$

According to Lemma 2.4.8, we have that $\beta^2 > k_2$. Thus, we only have the case

$$\begin{cases} a \geq \beta^2, \\ a \geq k_1. \end{cases}$$

Since $a \leq 2\sqrt{2}-2$, we have that $\beta \leq \sqrt{2\sqrt{2}-2}$. Again, from Lemma 2.4.8, we have $\beta^2 < k_1$. Thus, the condition is now changed into

$$k_1 \leq a \leq 2\sqrt{2}-2.$$

The condition of $x_1 \leq x_2 \leq \alpha < \beta$.
$x_1 \leq x_2 \leq \alpha < \beta$ is equivalent to

$$\sqrt{a + \frac{a\sqrt{4-4a-a^2}}{2}} \leq \alpha,$$

or

$$\begin{cases} a \leq \alpha^2, \\ a^4 + 4a^3 - 8a\alpha^2 + 4\alpha^4 \geq 0. \end{cases}$$

This is equivalent to

$$\begin{cases} a \leq \alpha^2, \\ a \geq h_1 \text{ or } a \leq h_2. \end{cases}$$

Combining with Lemma 2.4.8, we have

$$\left[\begin{array}{l} h_1 \leq a \leq \alpha^2, \\ a \leq h_2. \end{array} \right.$$

This means

$$\begin{cases} \text{for } \alpha \geq \sqrt{2\sqrt{2}-2} : 2\sqrt{2}-2 \geq a \geq h_1 \text{ or } a \leq h_2, \\ \text{for } \alpha < \sqrt{2\sqrt{2}-2} : a \leq h_2. \end{cases}$$

■

Lemma 2.4.13. *For $0 < a \leq 2\sqrt{2}-2$, we have that the following function*

$$G(a) = R(a, x_2(a)) = \exp(-x_2)\frac{2x_2^2 - 2x_2a + a^2}{2x_2^2 + 2x_2a + a^2}$$

is a decreasing function.

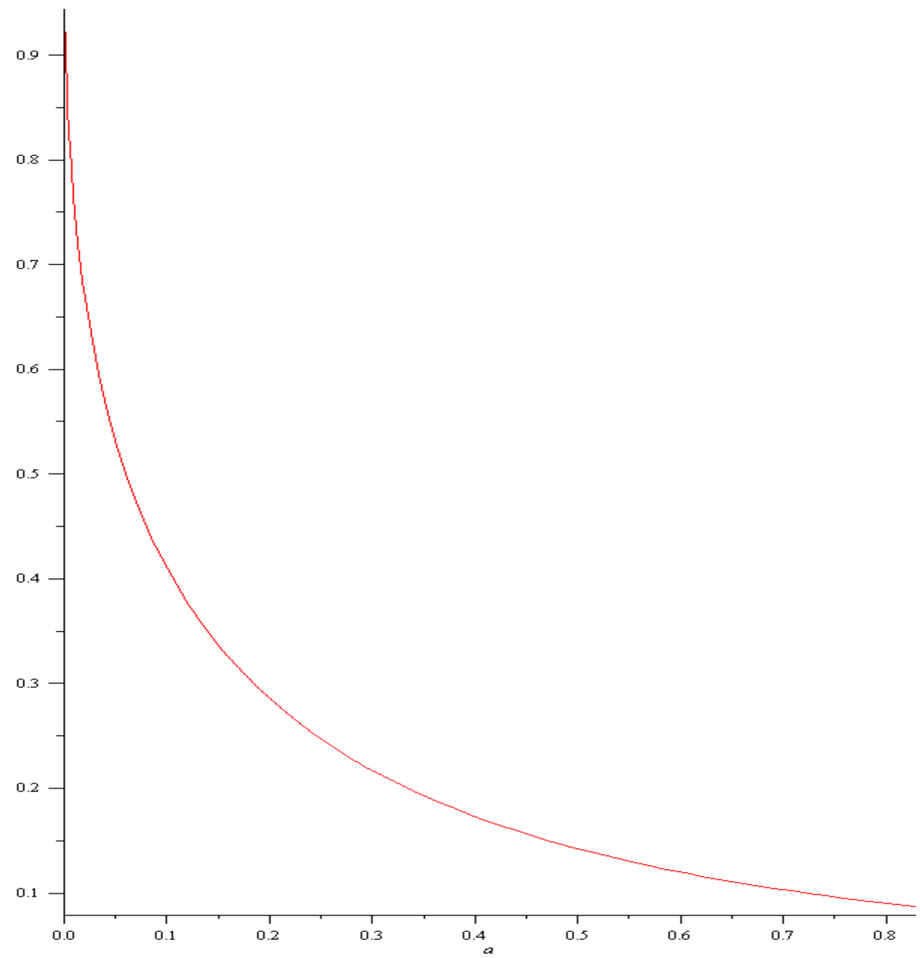


Figure 2.4.7. The graph of $G(a)$.

Lemma 2.4.14. *For $\beta \in [\sqrt{2\sqrt{2}-2}, 1]$, the function $k_1(\beta)$ is a decreasing function. For $\beta \in [0,1]$, the function $k_2(\beta)$ is an increasing function.*

Proof.

For $\beta \in [\sqrt{2\sqrt{2}-2}, 1]$, the function $k_1(\beta)$ is a decreasing function.

We have that

$$k_1 = -1 + \sqrt{1-\beta^2} + \sqrt{2+\beta^2-2\sqrt{1-\beta^2}}.$$

$$k_1' = -\frac{\beta(\sqrt{2+\beta^2-2\sqrt{1-\beta^2}} - \sqrt{1-\beta^2} - 1)}{\sqrt{2+\beta^2-2\sqrt{1-\beta^2}}\sqrt{1-\beta^2}}.$$

For $\beta \geq \sqrt{2\sqrt{2}-2}$, we have

$$\beta^4 + 4\beta^2 - 4 \geq 0.$$

Which means

$$\beta^2 \geq 2\sqrt{1-\beta^2}.$$

Thus

$$2 - \beta^2 + 2\sqrt{1-\beta^2} \leq 2 + \beta^2 - 2\sqrt{1-\beta^2},$$

or

$$(\sqrt{1-\beta^2}+1)^2 \leq 2 + \beta^2 - 2\sqrt{1-\beta^2}.$$

Therefore $k_1'(\beta) \leq 0$ and k_1 is a decreasing function.

For $\beta \in [0,1]$, the function $k_2(\beta)$ is an increasing function.

We have that

$$k_2 = -1 - \sqrt{1-\beta^2} + \sqrt{2+\beta^2+2\sqrt{1-\beta^2}}.$$

Then

$$k_2'(\beta) = \frac{\beta(\sqrt{2+\beta^2+2\sqrt{1-\beta^2}} + \sqrt{1-\beta^2} - 1)}{\sqrt{2+\beta^2+2\sqrt{1-\beta^2}}\sqrt{1-\beta^2}} > 0.$$

Thus k_2 is an increasing function.

∎

Lemma 2.4.15. *For* $\beta \geq \sqrt{2\sqrt{2}-2}$*, we have that* $R(k_1, \frac{k_1}{\sqrt{2}}) \geq R(k_1, \beta)$*. And for* $\beta \leq \sqrt{2\sqrt{2}-2}$*, we have that* $R(k_2, \frac{k_2}{\sqrt{2}}) \leq R(k_2, \beta)$*.*

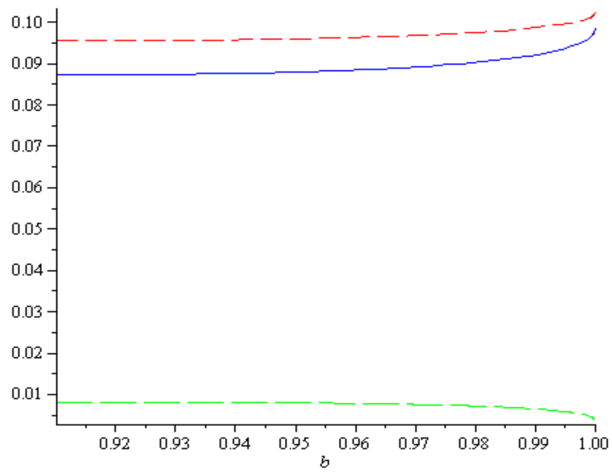


Figure 2.4.8. The graphs of $R(k_1, \frac{k_1}{\sqrt{2}})$ *(dash),* $R(k_1, \beta)$ *(line) and their subtraction (long dash).*

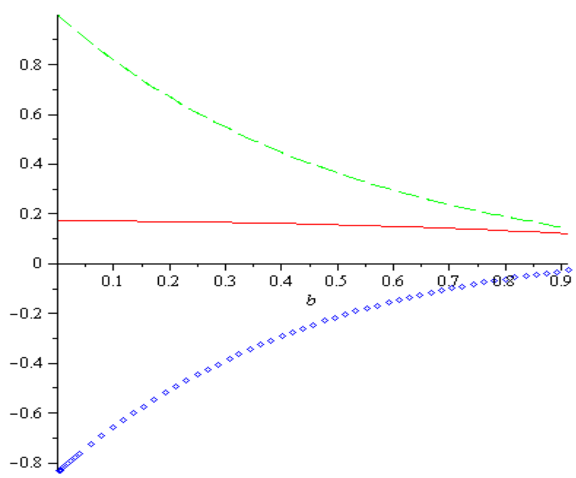


Figure 2.4.9. The graphs of $R(k_2, \frac{k_2}{\sqrt{2}})$ *(line),* $R(k_1, \beta)$ *(dash) and their subtraction (dot).*

Lemma 2.4.16. *For $a \geq 2\sqrt{2}-2$, there exists a unique solution $B(a)$ of $R(a, B(a)) = R(a, x_2(a))$. Then $B(a) \leq x_2(a)$ and $R(a,x) > R(a, B(a))$ iff $x < B(a)$.*

Proof. We have that

$$H_x(a,x) = -\exp(-x)\frac{4x^4 - 8ax^2 + a^4 + 4a^3}{(2x^2 + 2ax + a^2)^2} = -\exp(-x)\frac{(x^2 - X_1)(x^2 - X_2)}{(2x^2 + 2ax + a^2)^2}.$$

Thus the function $R(a,.)$ decreases in $[0, x_1]$ and $[x_2, +\infty)$ and increases in $[x_1, x_2]$. Since $R(a, 0) = 1$ and $R(a, x_2) < 1$, there exists $B(a)$ such that $R(a, B(a)) = R(a, x_2(a))$. In addition, we know that $R(a,.)$ decreases in $[x_2, +\infty)$; which implies that $B(a) \leq x_2(a)$.

From the increasing and decreasing property of R, we can see that $R(a,x) > R(a, B(a))$ iff $x < B(a)$.

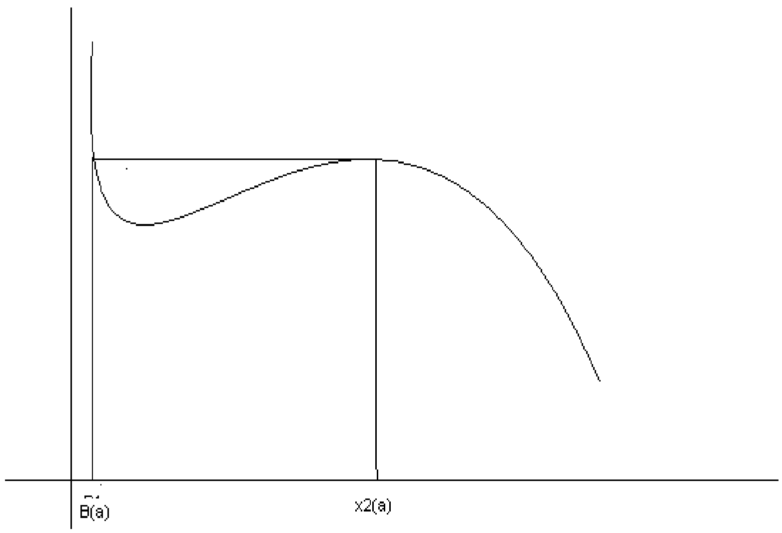


Figure 2.4.10. $B(a)$ and $x_2(a)$.

$\blacksquare$

Remark 2.4.2. *Lemma* 2.4.16 *deduces the existence of* B_1 *and* B_2*.*

Lemma 2.4.17. *For* $a > 0$*, the equation*

$$R(a, \alpha) = R(a, \beta) \text{ in } [\sqrt{2}\alpha, \sqrt{2}\beta].$$

has at most one solution. If the solution exists, we will call it S_1*.*

The equation

$$R(a, \alpha) = R(a, x_2) \text{ in } [\sqrt{2}\alpha, \infty).$$

has at most one solution. If the solution exists, we will call it S_2*.*

Proof.

According to Lemma 2.4.3, the function $R(., \alpha)$ increases in $[\sqrt{2}\alpha, \infty)$ and the function $R(., \beta)$ decreases in $(-\infty, \sqrt{2}\beta]$. Thus

$$R(a, \alpha) = R(a, \beta) \text{ in } [\sqrt{2}\alpha, \sqrt{2}\beta].$$

has at most one solution.

According to Lemmas 2.4.3 and 2.4.13, the function $R(., \alpha)$ increases in $[\sqrt{2}\alpha, \infty)$ and the function $R(a, x_2(a))$ decreases. Thus

$$R(a, \alpha) = R(a, x_2) \text{ in } [\sqrt{2}\alpha, \infty).$$

has at most one solution.

■

Lemma 2.4.18. *For* $0 < \beta \leq \sqrt{2\sqrt{2} - 2}$*, we have that*

$$\sqrt{1 - (\sqrt{4 - (1 - \sqrt{1 - \beta^2})^2} - 1)^2} > B_2.$$

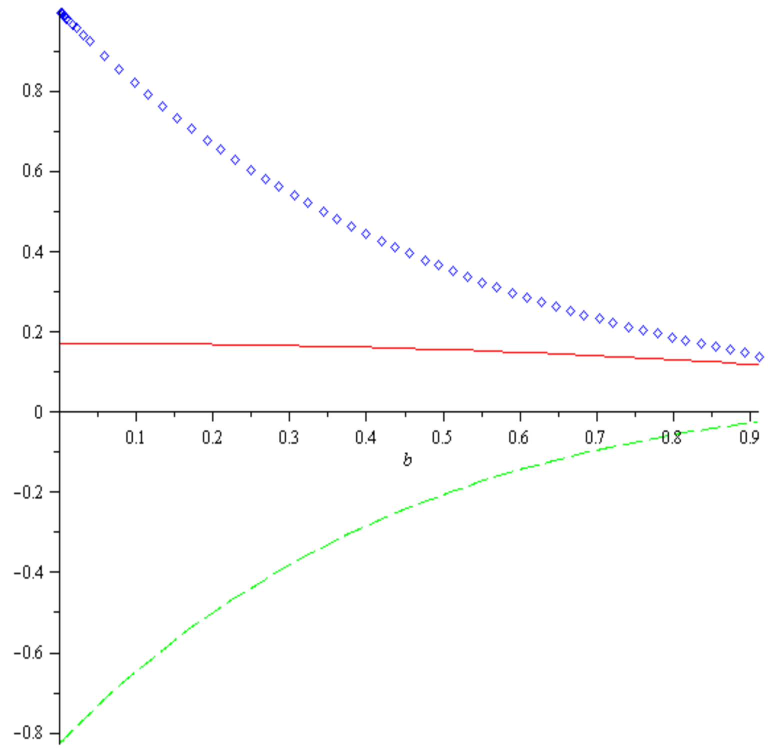


Figure 2.4.11. The graphs of $R(k_2, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2})$ (line), $R(k_2, \beta)$ (dot), and their substraction (dash).

$\blacksquare$

Proof of Theorem 2.4.1. We have

$$R(a,x) = \exp(-x)\frac{2x^2 - 2xa + a^2}{2x^2 + 2xa + a^2} =: f(x).$$

Differentiate f, we get

$$f'(x) = -\exp(-x)\frac{4x^4 - 8ax^2 + a^4 + 4a^3}{(2x^2 + 2ax + a^2)^2}. \tag{2.4.6}$$

We have

$$\Delta' = 16a^2 - 4a^4 - 16a^3 = 4a^2(4 - 4a - a^2) = 4a^2(a + 2 + 2\sqrt{2})(2\sqrt{2} - 2 - a). \tag{2.4.7}$$

* If $a \geq 2\sqrt{2} - 2$. Since $\Delta' \leq 0$, $f' \leq 0$. Since $x \in [L\sqrt{\frac{2\pi}{T\nu}}, L\sqrt{\frac{2\pi}{\triangle t\nu}}]$, we have that

$$f(x) \leq f(L\sqrt{\frac{2\pi}{\nu T}}) = \exp(-L\sqrt{\frac{2\pi}{\nu T}})\frac{(L\sqrt{\frac{2\pi}{\nu T}} - \frac{p}{\nu}L)^2 + L^2\frac{2\pi}{\nu T}}{(L\sqrt{\frac{2\pi}{\nu T}} + \frac{p}{\nu}L)^2 + L^2\frac{2\pi}{\nu T}}. \tag{2.4.8}$$

* If $a < 2\sqrt{2} - 2$. Notice that $\alpha_0 < 2\sqrt{2} - 2$.

We have

$$f'(x) = -\exp(-x)\frac{4x^4 - 8ax^2 + a^4 + 4a^3}{(2x^2 + 2ax + a^2)^2} = -\exp(-x)\frac{(x^2 - X_1)(x^2 - X_2)}{(2x^2 + 2ax + a^2)^2}, \tag{2.4.9}$$

where

$$X_1 = \frac{2a - a\sqrt{4 - 4a - a^2}}{2}, \tag{2.4.10}$$

and

$$X_2 = \frac{2a + a\sqrt{4 - 4a - a^2}}{2}. \tag{2.4.11}$$

We have the following table

a			x_1		x_2		
F'_y	-		0	+	0		-
F_y	1	$\searrow$		$\nearrow$		$\searrow$	

Since $\alpha = L\sqrt{\frac{2\pi}{T\nu}}$, $\beta = L\sqrt{\frac{2\pi}{\Delta t\nu}}$, then $x \in [\alpha, \beta]$.

Case 1: $\alpha < \beta \leq 1$

We put

$$h(a) = a^4 + 4a^3 - 8\alpha^2 a + 4\alpha^4, \tag{2.4.12}$$

and

$$k(a) = a^4 + 4a^3 - 8\beta^2 a + 4\beta^4. \tag{2.4.13}$$

Step 1: The properties of h and k

By Lemma 2.4.1, we can see that h has the following two positive solutions

$$h_1 = -1 + \sqrt{1-\alpha^2} + (2 + \alpha^2 - 2\sqrt{1-\alpha^2})^{\frac{1}{2}}, \tag{2.4.14}$$

and

$$h_2 = -1 - \sqrt{1-\alpha^2} + (2 + \alpha^2 + 2\sqrt{1-\alpha^2})^{\frac{1}{2}}. \tag{2.4.15}$$

And k has the following two positive solutions (can be coincided)

$$k_1 = -1 + \sqrt{1-\beta^2} + (2 + \beta^2 - 2\sqrt{1-\beta^2})^{\frac{1}{2}}, \tag{2.4.16}$$

and

$$k_2 = -1 - \sqrt{1-\beta^2} + (2 + \beta^2 + 2\sqrt{1-\beta^2})^{\frac{1}{2}}. \tag{2.4.17}$$

We have the following two tables according to Lemma 2.4.2,

a	0	α_1	β_1	$\alpha 0$	$2\sqrt{2}-2$
h'	-	0	$+$		
h	$1\searrow$		$\nearrow$		

and

a	0	α_1	β_1	$\alpha 0$	$2\sqrt{2}-2$
k'		-	0	$+$	
k	1	$\searrow$		$\nearrow$	

where $2\alpha = \alpha_1^3 + 3\alpha_1^2$ and $2\beta = \beta_1^3 + 3\beta_1^2$.

Therefore, if $a \in [h_2, h_1]$, we have that $a^4 + 4a^3 - 8\alpha^2 a + 4\alpha^4 \leq 0$. Which leads to

$$x_1 \leq \alpha \leq x_2. \tag{2.4.18}$$

Similarly, if $a \in [k_2, k_1]$, we have that $a^4 + 4a^3 - 8\beta^2 a + 4\beta^4 \leq 0$. Which leads to

$$x_1 \leq \beta \leq x_2. \tag{2.4.19}$$

Step 2: We will prove that $k_2 \geq h_2$

We have that $k_2 \geq h_2$ is equivalent to

$$-1-\sqrt{1-\beta^2}+(2+\beta^2+2\sqrt{1-\beta^2})^{\frac{1}{2}} \geq -1-\sqrt{1-\alpha^2}+(2+\alpha^2+2\sqrt{1-\alpha^2})^{\frac{1}{2}}. \tag{2.4.20}$$

This is equivalent to

$$\frac{1-\sqrt{1-\beta^2}}{2}+(1-(\frac{1-\sqrt{1-\beta^2}}{2})^2)^{\frac{1}{2}} \geq \frac{1-\sqrt{1-\alpha^2}}{2}+(1-(\frac{1-\sqrt{1-\alpha^2}}{2})^2)^{\frac{1}{2}}. \tag{2.4.21}$$

We put $\sin\theta_1 = \frac{1-\sqrt{1-\beta^2}}{2}$ and $\sin\theta_2 = \frac{1-\sqrt{1-\alpha^2}}{2}$. Then (2.4.21) is equivalent to $0 \leq \theta_2 \leq \theta_1 \leq \frac{\pi}{2}$ and

$$\sin\theta_1 + \cos\theta_1 \geq \sin\theta_2 + \cos\theta_2. \tag{2.4.22}$$

And (2.4.22) is equivalent to

$$\sin\frac{\theta_1-\theta_2}{2}\cos(\frac{\theta_1+\theta_2}{2}+\frac{\pi}{4}) \geq 0. \tag{2.4.23}$$

And (2.4.23) is now equivalent to

$$\frac{\pi}{4} \leq \frac{\theta_1+\theta_2}{2}+\frac{\pi}{4} \leq \frac{\pi}{2}. \tag{2.4.24}$$

And this is again equivalent to

$$0 \leq \theta_1+\theta_2 \leq \frac{\pi}{2}, \tag{2.4.25}$$

or

$$\sin\theta_2 \leq \cos\theta_1, \tag{2.4.26}$$

or

$$\frac{1-\sqrt{1-\alpha^2}}{2} \leq (1-(\frac{1-\sqrt{1-\beta^2}}{2})^2)^{\frac{1}{2}}. \tag{2.4.27}$$

(2.4.27) is correct because from

$$-\alpha^2 - 2\sqrt{1-\alpha^2} - \beta^2 - 2\sqrt{1-\beta^2} < 0, \tag{2.4.28}$$

we can get

$$(\frac{1-\sqrt{1-\beta^2}}{2})^2 + (\frac{1-\sqrt{1-\alpha^2}}{2})^2 \leq 1. \tag{2.4.29}$$

Step 3: We prove that

If $-\alpha^2 + 2\sqrt{1-\alpha^2} \leq \beta^2 - 2\sqrt{1-\beta^2}$, *then* $h_1 \geq k_1$.

If $-\alpha^2 + 2\sqrt{1-\alpha^2} > \beta^2 - 2\sqrt{1-\beta^2}$, *then* $h_1 < k_1$.

Suppose that $-\alpha^2 + 2\sqrt{1-\alpha^2} \leq \beta^2 - 2\sqrt{1-\beta^2}$ (the other case is proven similarly). He have

$$(1+\sqrt{1-\alpha^2})^2 \leq 4 - (1+\sqrt{1-\beta^2})^2. \tag{2.4.30}$$

Thus

$$\frac{1+\sqrt{1-\alpha^2}}{2} \leq (1-(\frac{1+\sqrt{1-\beta^2}}{2})^2)^{\frac{1}{2}}. \tag{2.4.31}$$

We put $\sin\theta_1 = \frac{1+\sqrt{1-\alpha^2}}{2}$ and $\cos\theta_2 = (1-(\frac{1+\sqrt{1-\beta^2}}{2})^2)^{\frac{1}{2}}$, where $0 \leq \theta_2 \leq \theta_1 \leq \frac{\pi}{2}$.

We have

$$\sin\theta_1 \leq \sin(\frac{\pi}{2} - \theta_2). \tag{2.4.32}$$

This implies

$$0 \leq \theta_1 + \theta_2 \leq \frac{\pi}{2}. \tag{2.4.33}$$

Hence

$$\sin\frac{\theta_1 - \theta_2}{2}\cos(\frac{\theta_1+\theta_2}{2} + \frac{\pi}{4}) \geq 0. \tag{2.4.34}$$

Therefore:

$$\sin\theta_1 + \cos\theta_1 \geq \sin\theta_2 + \cos\theta_2. \tag{2.4.35}$$

Thus $h_1 \geq k_1$.

Step 4: We will prove that

If $-\alpha^2 + 2\sqrt{1-\alpha^2} \leq \beta^2 + 2\sqrt{1-\beta^2}$, *we have* $k_2 \leq h_1$.

If $-\alpha^2 + 2\sqrt{1-\alpha} > \beta^2 + 2\sqrt{1-\beta^2}$, *we have* $k_2 > h_1$.

Suppose that $-\alpha^2 + 2\sqrt{1-\alpha^2} \leq \beta^2 + 2\sqrt{1-\beta^2}$, we prove that $k_2 \leq h_1$ (the other case is proven similarly). We have

$$(\frac{1-\sqrt{1-\beta^2}}{2})^2 \leq 1 - (\frac{1+\sqrt{1-\alpha^2}}{2})^2. \tag{2.4.36}$$

Put $\sin\theta_1 = \frac{1-\sqrt{1-\beta^2}}{2}$ and $\sin\theta_2 = \frac{1+\sqrt{1-\alpha^2}}{2}$, where $0 \leq \theta_1 \leq \theta_2 \leq \frac{\pi}{2}$. Therefore

$$\sin\theta_1 + \cos\theta_1 \leq \sin\theta_2 + \cos\theta_2. \tag{2.4.37}$$

From (2.4.37), we can get that $k_2 \leq h_1$.

Step 5: The cases

Case 1a $\alpha \geq \sqrt{2\sqrt{2}-2}$

From Lemma 2.4.12, we have that for $h_1 \leq a \leq 2\sqrt{2}-2$ and $a \leq h_2$, $x_1 \leq x_2 \leq \alpha \leq \beta$ or $\max\limits_x R(a,x) = R(a,\alpha)$. Moreover for $a \geq 2\sqrt{2}-2$, $\max\limits_x R(a,x) = R(a,\alpha)$. Thus for $a \geq h_1$ or $a \leq h_2$, $\max\limits_x R(a,x) = R(a,\alpha)$. Moreover we always have that $\max\limits_x R(a,x) \geq R(a,\alpha)$. According to Lemma 2.4.3 $R(a,\alpha) \geq R(\sqrt{2}\alpha,\alpha)$ and according to Lemma 2.4.10 $h_1 \leq \sqrt{2}\alpha$. Hence $\min\limits_a \max\limits_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

Case 1b $\alpha < \sqrt{2\sqrt{2}-2}$

Case 1b1 $-\alpha^2 + 2\sqrt{1-\alpha^2} \leq \beta^2 - 2\sqrt{1-\beta^2}$ or $(\sqrt{1-\alpha^2}+1)^2 + (1+\sqrt{1-\beta^2})^2 \leq 4$

This condition leads to $h_2 \leq k_2 \leq k_1 \leq h_1$.

Since $\alpha < \beta$, we have that $(1+\sqrt{1-\beta^2})^2 < 2$. Which implies $\beta > \sqrt{2\sqrt{2}-2}$.

We have the following tables

a		h_2	k_2		k_1	h_1	
h	+	0		-		0	+

and

a		h_2	k_2		k_1	h_1	
k	+		0	-	0		+

If $a \geq 2\sqrt{2}-2$, we have that $\max_x R(a,x) = R(a,\alpha)$.

If $h_1 \leq a < 2\sqrt{2}-2$, we have $k(a) > 0$ and $h(a) > 0$. Thus $\alpha \leq x_1 \leq x_2 \leq \beta$. Hence $\max_x R(a,x) = \max\{R(a,\alpha), R(a,x_2)\}$.

If $k_1 \leq a < h_1$, we have $h(a) \leq 0$ and $k(a) \geq 0$. Thus $x_1 \leq \alpha \leq x_2 \leq \beta$. Hence $\max_x R(a,x) = R(a,x_2)$.

If $k_2 \leq a < k_1$, we have $h(a) \leq 0$ and $k(a) \leq 0$. Thus $x_1 \leq \alpha \leq \beta \leq x_2$. Hence $\max_x R(a,x) = R(a,\beta)$.

If $h_2 \leq a < k_2$, we have $h(a) \leq 0$ and $k(a) \geq 0$. Thus $x_1 \leq \alpha \leq x_2 \leq \beta$. Hence $\max_x R(a,x) = R(a,x_2)$.

If $a < h_2$, we have $x_1 \leq x_2 \leq \alpha \leq \beta$. Hence $\max_x R(a,x) = R(a,\alpha)$.

We consider two cases

If $\alpha \geq M_0$. From Lemma 2.4.10 we get $h_1 \leq \sqrt{2}\alpha$. If $\sqrt{2}\alpha < 2\sqrt{2}-2$, according to Lemma 2.4.4 we have $R(\sqrt{2}\alpha, x_2) \leq R(\sqrt{2}\alpha,\alpha)$. Thus $\max_x R(\sqrt{2}\alpha, x) = R(\sqrt{2}\alpha,\alpha)$ for $\sqrt{2}\alpha < 2\sqrt{2}-2$ and $\sqrt{2}\alpha \geq 2\sqrt{2}-2$. Moreover $\max_x R(a,x) \geq R(a,\alpha) \geq R(\sqrt{2}\alpha,\alpha)$. Thus $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < M_0$, from $(\sqrt{1-\alpha^2}+1)^2 + (1+\sqrt{1-\beta^2})^2 \leq 4$, we have that

$$(\sqrt{1-M_0^2}+1)^2 + (1+\sqrt{1-\beta^2})^2 \leq 4.$$

Thus

$$(1+\sqrt{1-\beta^2})^2 \leq 4 - (\sqrt{1-M_0^2}+1)^2.$$

However, we have that $4-(\sqrt{1-M_0^2}+1)^2 < 4-(\sqrt{1-0.49^2}+1)^2 = 0.496655 < 1$. Which implies

$$(1+\sqrt{1-\beta^2})^2 < 1.$$

This is a contradiction. Thus this case does not occur.

Case 1b2: $\beta^2 + 2\sqrt{1-\beta^2} \geq -\alpha^2 + 2\sqrt{1-\alpha^2} > \beta^2 - 2\sqrt{1-\beta^2}$, or $(1+\sqrt{1-\alpha^2})^2 + (1-\sqrt{1-\beta^2})^2 \leq 4 \leq (1+\sqrt{1-\alpha^2})^2 + (1+\sqrt{1-\beta^2})^2$.

This condition leads to $h_2 \le k_2 \le h_1 \le k_1$.

If $\alpha < \beta \le \sqrt{2\sqrt{2}-2}$

If $a \ge 2\sqrt{2}-2$, we have that $\max_x R(a,x) = R(a,\alpha)$.

If $k_1 \le a < 2\sqrt{2}-2$, $\alpha < \beta \le x_1 \le x_2$. Hence $\max_x R(a,x) = R(a,\alpha)$.

If $h_1 \le a < k_1$, we have $h(a) \ge 0$ and $k(a) \le 0$. Thus $\alpha \le x_1 \le \beta \le x_2$. Hence $\max_x R(a,x) = \max\{R(a,\alpha), R(a,\beta)\}$.

If $k_2 \le a < h_1$, we have $h(a) \le 0$ and $k(a) \le 0$. Thus $x_1 \le \alpha \le \beta \le x_2$. Hence $\max_x R(a,x) = R(a,\beta)$.

If $h_2 \le a < k_2$, we have $h(a) \le 0$ and $k(a) \ge 0$. Thus $x_1 \le \alpha \le x_2 \le \beta$. Hence $\max_x R(a,x) = R(a,x_2)$.

If $a < h_2$, we have $x_1 \le x_2 \le \alpha \le \beta$. Hence $\max_x R(a,x) = R(a,\alpha)$.

According to Lemma 2.4.10: $h_1 \le \sqrt{2}\alpha$. We consider the following cases

*If $h_1 \le \sqrt{2}\alpha \le k_1$, then according to Lemma 2.4.5, $\alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$.

According to Lemma 2.4.6, if $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})) \le \alpha$, we have that $R(\sqrt{2}\alpha,\beta) \le R(\sqrt{2}\alpha,\alpha)$. Then $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$. Since $\beta \le 1$, we have $\alpha \le \frac{1}{k}$. According to Lemma 2.4.7, we have that $\frac{2\sqrt{2}(k^2-1)}{k^4+1} \le \frac{1}{k}$ and $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})) \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$ if $k \le M_1$. Which means if $k \le M_1$ and $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})) \le \alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$, we have that $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < \frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2}))$ and $k \le M_1$ or $k > M_1$ and $\alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$, then $R(\sqrt{2}\alpha,\beta) > R(\sqrt{2}\alpha,\alpha)$. Moreover, when $a = k_1$, we have that $a^4+4a^3-8\beta^2a+4\beta^4 = 0$. Thus $\beta = x_1(k_1)$ or $x_2(k_1)$. From Lemma 2.4.8, we have that $\beta^2 \le k_1$. Thus $\beta = x_1(k_1)$, which means $R(k_1,\alpha) \ge R(k_1,\beta)$. Therefore there exists a solution S_1 in $[\sqrt{2}\alpha, k_1]$ of $R(S_1,\alpha) = R(S_1,\beta)$ and $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

*If $k_1 < \sqrt{2}\alpha \le 2\sqrt{2}-2$, then according to Lemma 2.4.5, $\alpha > \frac{2\sqrt{2}(k^2-1)}{k^4+1}$. We can easily see that $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < \sqrt{2\sqrt{2}-2} < \beta$

If $a \ge 2\sqrt{2}-2$, we have that $\max_x R(a,x) = R(a,\alpha)$.

If $k_1 \le a < 2\sqrt{2}-2$, $\alpha \le x_1 \le x_2 \le \beta$. Hence $\max_x R(a,x) = \max\{R(a,\alpha), R(a,x_2)\}$.

If $h_1 \le a < k_1$, we have $h(a) \ge 0$ and $k(a) \le 0$. Thus $\alpha \le x_1 \le \beta \le x_2$. Hence $\max_x R(a,x) = \max\{R(a,\alpha), R(a,\beta)\}$.

If $k_2 \le a < h_1$, we have $h(a) \le 0$ and $k(a) \le 0$. Thus $x_1 \le \alpha \le \beta \le x_2$. Hence $\max_x R(a,x) = R(a,\beta)$.

If $h_2 \le a < k_2$, we have $h(a) \le 0$ and $k(a) \ge 0$. Thus $x_1 \le \alpha \le x_2 \le \beta$. Hence $\max_x R(a,x) = R(a,x_2)$.

If $a < h_2$, we have $x_1 \le x_2 \le \alpha \le \beta$. Hence $\max_x R(a,x) = R(a,\alpha)$.

According to Lemma 2.4.10: $h_1 \le \sqrt{2}\alpha$. We consider the following cases

*If $h_1 \le \sqrt{2}\alpha \le k_1$, then according to Lemma 2.4.5, $\alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$.

According to Lemma 2.4.6, if $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})) \le \alpha$, we have that $R(\sqrt{2}\alpha,\beta) \le R(\sqrt{2}\alpha,\alpha)$. Then $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$. Since $\beta \le 1$, we have $\alpha \le \frac{1}{k}$. According to Lemma 2.4.7, we have that $\frac{2\sqrt{2}(k^2-1)}{k^4+1} \le \frac{1}{k}$ and $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})) \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$ if $k \le M_1$. Which means if $k \le M_1$ and $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})) \le \alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$, we have that $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < \frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2}))$ and $k \le M_1$ or $k > M_1$, then $R(\sqrt{2}\alpha,\beta) > R(\sqrt{2}\alpha,\alpha)$. Moreover, when $a = k_1$, we have that $a^4 + 4a^3 - 8\beta^2 a + 4\beta^4 = 0$. Thus $\beta = x_1(k_1)$ or $x_2(k_1)$. From Lemma 2.4.8, we have that $\beta^2 \ge k_1$. Thus $\beta = x_2(k_1)$. From Lemma 2.4.15, we have that $R(k_1, \frac{k_1}{\sqrt{2}}) \ge R(k_1,\beta)$. Hence $B_1 \ge \frac{k_1}{\sqrt{2}} \ge \alpha$ according to Lemma 2.4.16, which means $R(k_1,\alpha) \ge R(k_1,\beta)$. Therefore there exists a solution S_1 in $[\sqrt{2}\alpha, k_1]$ of $R(S_1,\alpha) = R(S_1,\beta)$ and $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

*If $k_1 < \sqrt{2}\alpha$, then according to Lemma 2.4.14, $\alpha > \frac{k_1}{\sqrt{2}} \ge \frac{k_1(1)}{\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{2}} > M_0$. Using Lemma 2.4.3, we can easily see that $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

Case 1b3: $\beta^2 + 2\sqrt{1-\beta^2} < -\alpha^2 + 2\sqrt{1-\alpha^2}$, or $(1+\sqrt{1-\alpha^2})^2 + (1-\sqrt{1-\beta^2})^2 > 4$ or $\alpha < \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}$.

This condition leads to $h_2 \le h_1 \le k_2 \le k_1$.

If $\alpha < \beta \le \sqrt{2\sqrt{2}-2}$

If $a \ge 2\sqrt{2}-2$, we have that $\max_x R(a,x) = R(a,\alpha)$.

If $k_1 \le a < 2\sqrt{2}-2$, $\alpha \le \beta \le x_1 \le x_2$. Hence $\max\limits_x R(a,x) = R(a,\alpha)$.

If $k_2 \le a < k_1$, we have $h(a) \ge 0$ and $k(a) \le 0$. Thus $\alpha \le x_1 \le \beta \le x_2$. Hence $\max\limits_x R(a,x) = \max\{R(a,\alpha), R(a,\beta)\}$.

If $h_1 \le a < k_2$, we have $h(a) \ge 0$ and $k(a) \ge 0$. Thus $\alpha \le x_1 \le x_2 \le \beta$. Hence $\max\limits_x R(a,x) = \max\{R(a,\alpha), R(a,x_2)\}$.

If $h_2 \le a < h_1$, we have $h(a) \le 0$ and $k(a) \ge 0$. Thus $x_1 \le \alpha \le x_2 \le \beta$. Hence $\max\limits_x R(a,x) = R(a,x_2)$.

If $a < h_2$, we have $x_1 \le x_2 \le \alpha \le \beta$. Hence $\max\limits_x R(a,x) = R(a,\alpha)$.

According to Lemma 2.4.10: $h_1 \le \sqrt{2}\alpha$, we consider the following three cases:

* If $\sqrt{2}\alpha \ge k_1$, we have that $\max\limits_x R(a,x) = R(a,\alpha)$. Thus $\min\limits_a \max\limits_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

* If $k_2 \le \sqrt{2}\alpha < k_1$, or $\alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$.

If $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2}) \le \alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$ and $k \le M_1$, we have that $R(\sqrt{2}\alpha,\beta) \le R(\sqrt{2}\alpha,\alpha)$. Thus $\min\limits_a \max\limits_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < \frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})$ and $k \le M_1$ or $k > M_1$, we have that $R(\sqrt{2}\alpha,\beta) > R(\sqrt{2}\alpha,\alpha)$. When $a = k_1$, we can see that $\beta^2 \le k_1$ from Lemma 2.4.8. Thus $\beta = x_1(k_1)$ and $R(k_1,\alpha) > R(k_1,\beta)$. Hence there is a solution S_1 in $[\sqrt{2}\alpha, k_1]$ of $R(S_1,\alpha) = R(S_1,\beta)$ and we have $\min\limits_a \max\limits_x R(a,x) = R(S_1,\alpha)$.

* If $\sqrt{2}\alpha < k_2$

If $\alpha \ge M_0$, then according to Lemma 2.4.4, we can see that $R(\sqrt{2}\alpha,\alpha) \ge R(\sqrt{2}\alpha,x_2)$. Which implies $\min\limits_a \max\limits_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < M_0$ then according to Lemma 2.4.4, we can see that $R(\sqrt{2}\alpha,\alpha) < R(\sqrt{2}\alpha,x_2)$. When $a = k_2$, we have that $\beta = x_2$ because of Lemma 2.4.8. If $\alpha < B_2$, using Lemma 2.4.16, we have $R(k_2,\beta) < R(k_2,\alpha)$. Thus there exists a solution S_2 of $R(S_2,\alpha) = R(S_2,x_2)$ and $\min\limits_a \max\limits_x R(a,x) = R(S_2,\alpha)$. If $\alpha \ge B_2$, then $R(k_2,\beta) \ge R(k_2,\alpha)$. When $a = k_1$, from Lemma 2.4.8, we can see that $\beta^2 \le k_1$. Thus $\beta = x_1(k_1)$. Therefore $R(k_1,\beta) < R(k_1,\alpha)$. Hence there exists a solution S_1 of $R(S_1,\beta) = R(S_1,\alpha)$ and $\min\limits_a \max\limits_x R(a,x) = R(S_1,\alpha)$.

If $\alpha < \sqrt{2\sqrt{2}-2} < \beta$

If $a \ge 2\sqrt{2}-2$, we have that $\max\limits_x R(a,x) = R(a,\alpha)$.

If $k_1 \le a < 2\sqrt{2}-2$, $\alpha \le x_1 \le x_2 \le \beta$. Hence $\max\limits_x R(a,x) =$

$\max\{R(a,\alpha), R(a,x_2)\}$.

If $k_2 \le a < k_1$, we have $h(a) \ge 0$ and $k(a) \le 0$. Thus $\alpha \le x_1 \le \beta \le x_2$. Hence $\max_x R(a,x) = \max\{R(a,\alpha), R(a,\beta)\}$.

If $h_1 \le a < k_2$, we have $h(a) \ge 0$ and $k(a) \ge 0$. Thus $\alpha \le x_1 \le x_2 \le \beta$. Hence $\max_x R(a,x) = \max\{R(a,\alpha), R(a,x_2)\}$.

If $h_2 \le a < h_1$, we have $h(a) \le 0$ and $k(a) \ge 0$. Thus $x_1 \le \alpha \le x_2 \le \beta$. Hence $\max_x R(a,x) = R(a,x_2)$.

If $a < h_2$, we have $x_1 \le x_2 \le \alpha \le \beta$. Hence $\max_x R(a,x) = R(a,\alpha)$.

According to Lemma 2.4.10: $h_1 \le \sqrt{2}\alpha$, we consider the following cases:

* If $\sqrt{2}\alpha \ge 2\sqrt{2}-2$, we have that $\max_x R(a,x) = R(a,\alpha)$. Thus $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

* If $k_2 \le \sqrt{2}\alpha \le k_1$, or $\alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$.

If $\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2}) \le \alpha \le \frac{2\sqrt{2}(k^2-1)}{k^4+1}$ and $k \le M_1$, we have that $R(\sqrt{2}\alpha,\beta) \le R(\sqrt{2}\alpha,\alpha)$. Thus $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < \frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})$ and $k \le M_1$ or $k > M_1$, we have that $R(\sqrt{2}\alpha,\beta) > R(\sqrt{2}\alpha,\alpha)$. When $a = k_1$, we can see that $\beta^2 > k_1$ from Lemma 2.4.8, thus $\beta = x_2(k_1)$. From Lemma 2.4.15 we have that $R(k_1,\frac{k_1}{\sqrt{2}}) \ge R(k_1,\beta)$, which means $\frac{k_1}{\sqrt{2}} \le B_1$ according to Lemma 2.4.16. Thus $\alpha \le B_1$, then $R(k_1,\alpha) > R(k_1,x_2(k_1))$. Hence there is a solution S_1 in $[\sqrt{2}\alpha, k_1]$ of $R(S_1,\alpha) = R(S_1,\beta)$ and we have $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

* If $\sqrt{2}\alpha < k_2$ or $\sqrt{2}\alpha > k_1$

If $\alpha \ge M_0$, then according to Lemma 2.4.4, we can see that $R(\sqrt{2}\alpha,\alpha) \ge R(\sqrt{2}\alpha,x_2)$. Which implies $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha < M_0$ (in this case we cannot have the condition $\sqrt{2}\alpha > k_1$ because from Lemma 2.4.14, we have $k_1 \ge k_1(1) = \sqrt{3}-1$ and $\frac{\sqrt{3}-1}{\sqrt{2}} > M_0$). When $a = k_2$, we have that $\beta = x_2$ because of Lemma 2.4.8. If $\alpha < B_2$, using Lemma 2.4.16, we have $R(k_2,\beta) < R(k_2,\alpha)$. Thus there exists a solution S_2 of $R(S_2,\alpha) = R(S_2,x_2)$ and $\min_a \max_x R(a,x) = R(S_2,\alpha)$. If $\alpha \ge B_2$, then $R(k_2,\beta) \ge R(k_2,\alpha)$. When $a = k_1$, from Lemma 2.4.8, we can see that $\beta^2 \ge k_1$. Thus $\beta = x_2(k_1)$. From Lemma 2.4.15 we have that $R(k_1,\frac{k_1}{\sqrt{2}}) \ge R(k_1,\beta)$, which means $\frac{k_1}{\sqrt{2}} \le B_1$ according to Lemma 2.4.16. Thus $\alpha < B_1$, then $R(k_1,\alpha) > R(k_1,x_2(k_1))$. Hence there is a solution S_1 in $[\sqrt{2}\alpha, k_1]$ of $R(S_1,\alpha) = R(S_1,\beta)$ and we have $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

Combining all of the cases above, we have that

If $\alpha < \beta \le 1$ we have the following disjoint cases

* If $\alpha < \min\{\sqrt{2\sqrt{2}-2}, \frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})), \frac{2\sqrt{2}(k^2-1)}{k^4+1}\}$, $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

According to Lemma 2.4.7, we have that $\frac{2\sqrt{2}(k^2-1)}{k^4+1} < \sqrt{2\sqrt{2}-2}$, thus: If $\alpha < \min\{\frac{1}{k-1}\ln(\frac{k^2-\sqrt{2}k+1}{k^2+\sqrt{2}k+1}(3+2\sqrt{2})), \frac{2\sqrt{2}(k^2-1)}{k^4+1}\}$, $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

* If $\alpha < \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}$, $\beta \le \sqrt{2\sqrt{2}-2}$ and $\alpha < \min\{\frac{k_2}{\sqrt{2}}, B_2, M_0\}$ $\min_a \max_x R(a,x) = R(S_2,\alpha)$.

According to Lemma 2.4.15, 2.4.16 and 2.4.18, we can see that $B_2 < \frac{k_2}{\sqrt{2}}$ and $B_2 < \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}$. Moreover from Lemma 2.4.14, we have that $\frac{k_2}{\sqrt{2}} \le \frac{k_2(\sqrt{2\sqrt{2}-2})}{\sqrt{2}} = 0.3521934495 < M_0$ thus: If $\beta \le \sqrt{2\sqrt{2}-2}$ and $\alpha < B_2$ $\min_a \max_x R(a,x) = R(S_2,\alpha)$.

* If $\alpha < \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}$, $\beta \le \sqrt{2\sqrt{2}-2}$ and $B_2 \le \alpha < \min\{\frac{k_2}{\sqrt{2}}, M_0\}$ $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

From Lemma 2.4.14, we have that $\frac{k_2}{\sqrt{2}} \le \frac{k_2(\sqrt{2\sqrt{2}-2})}{\sqrt{2}} = 0.3521934495 < M_0$ thus: If $\beta \le \sqrt{2\sqrt{2}-2}$ and $B_2 \le \alpha < \min\{\frac{k_2}{\sqrt{2}}, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

* If $\beta > \sqrt{2\sqrt{2}-2}$ and $\alpha < \min\{\frac{k_2}{\sqrt{2}}, B_2, M_0, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_2,\alpha)$.

* If $\beta > \sqrt{2\sqrt{2}-2}$ and $B_2 \le \alpha < \min\{\frac{k_2}{\sqrt{2}}, M_0, \sqrt{1-(\sqrt{4-(1-\sqrt{1-\beta^2})^2}-1)^2}\}$ $\min_a \max_x R(a,x) = R(S_1,\alpha)$.

* Otherwise $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

Case 2: $\alpha \le 1 < \beta$

*If $\alpha \ge \sqrt{2-2\sqrt{2}}$, then for $a \ge \sqrt{2\sqrt{2}-2}$, $\sqrt{2\sqrt{2}-2} \ge a \ge h_1$ and $a \le h_2$ we have that $\max_x R(a,x) = R(a,\alpha)$. Thus $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

*If $\alpha < \sqrt{2-2\sqrt{2}}$.

We can see that in this case $k(a) > 0$ for all positive a.

For $a \geq 2\sqrt{2}-2$ $\max_x R(a,x) = R(a,\alpha)$.

For $a \leq 2\sqrt{2}-2 < 1 < \beta$. If $a < h_2$, $x_1 \leq x_2 \leq \alpha \leq \beta$. Thus $\max_x R(a,x) = R(a,\alpha)$. If $a > h_1$, we have that $\alpha \leq x_1 \leq x_2 \leq \beta$, then $\max_x R(a,x) = \max\{R(a,\alpha), R(a,x_2)\}$.

We consider two cases:

If $\alpha > M_0$, using Lemma 2.4.4, we have that $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

If $\alpha \leq M_0$, using Lemmas 2.4.4, 2.4.11, the equation $R(a,\alpha) = R(a,x_2)$ has a solution S_2 in $[\sqrt{2}\alpha, 2\sqrt{2}-2]$ and $\min_a \max_x R(a,x) = R(S_2,\alpha)$.

Case 3: $1 < \alpha < \beta$

*If $\alpha \geq \sqrt{2-2\sqrt{2}}$, then for $a \geq \sqrt{2\sqrt{2}-2}$, $\sqrt{2\sqrt{2}-2} \geq a \geq h_1$ and $a \leq h_2$ we have that $\max_x R(a,x) = R(a,\alpha)$. Thus $\min_a \max_x R(a,x) = R(\sqrt{2}\alpha,\alpha)$.

*If $\alpha < \sqrt{2-2\sqrt{2}}$.

We can see that in this case $k(a) > 0$ and $h(a) > 0$ for all postive a.

For $a \geq 2\sqrt{2}-2$ $\max_x R(a,x) = R(a,\alpha)$.

For $a \leq 2\sqrt{2}-2 < 1 < \alpha^2 < \beta^2$. Thus $x_1 \leq x_2 \leq \alpha < \beta$, then $\max_x R(a,x) = R(a,\alpha)$.

$$\min_a \max_\omega \rho = \exp(-\sqrt{\alpha})(3-2\sqrt{2})$$

when (ω, a) is $(\alpha^2 \frac{\nu}{2L^2}, \sqrt{2}\alpha)$.

Chapter 3

Optimized Schwarz Waveform Relaxation Methods For The Two Dimensional Heat Equation

3.1 Optimized Schwarz Waveform Relaxation Methods For The Two Dimensional Heat Equation With Robin Transmission Condition

In this section, we are interested in the following heat equation:

$$\begin{cases} \mathfrak{L}u = \partial_t u - \nu\partial_{xx}u - \nu\partial_{yy}u = f & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.1.1}$$

We consider the following algorithm

$$\begin{cases} \mathfrak{L}u_1^k = f & \text{in } (-\infty, L) \times \mathbb{R} \times (0,T), \\ u_1^k(x,y,0) = u_0(x,y) & \text{in } (-\infty, L) \times \mathbb{R}, \\ (\partial_x + \frac{p}{2\nu})u_1^k(L,.,.) = (\partial_x + \frac{p}{2\nu})u_2^{k-1}(L,.,.) & \text{in } \mathbb{R} \times (0,T), \end{cases} \tag{3.1.2}$$

$$\begin{cases} \mathfrak{L}u_2^k = f & \text{in } (0,\infty)\times\mathbb{R}\times(0,T),\\ u_2^k(x,y,0) = u_0(x,y) & \text{in } (0,\infty)\times\mathbb{R},\\ (\partial_x - \frac{p}{2\nu})u_2^k(0,.,.) = (\partial_x - \frac{p}{2\nu})u_1^{k-1}(0,.,.) & \text{in } \mathbb{R}\times(0,T). \end{cases}$$

We consider the algorithm (3.1.2) and put $e_1^k = u_1^k - u$, $e_2^k = u_2^k - u$ where u is the solution of the equation (3.1.1), then we have that

$$\begin{cases} \mathfrak{L}e_1^k = 0 & \text{in } (-\infty,L)\times\mathbb{R}\times(0,T),\\ e_1^k(x,y,0) = 0 & \text{in } (-\infty,L)\times\mathbb{R},\\ (\partial_x + \frac{p}{2\nu})e_1^k(L,.,.) = (\partial_x + \frac{p}{2\nu})e_2^{k-1}(L,.,.) = h_L^{k-1} & \text{in } \mathbb{R}\times(0,T), \end{cases} \tag{3.1.3}$$

$$\begin{cases} \mathfrak{L}e_2^k = 0 & \text{in } (0,\infty)\times\mathbb{R}\times(0,T),\\ e_2^k(x,y,0) = 0 & \text{in } (0,\infty)\times\mathbb{R},\\ (\partial_x - \frac{p}{2\nu})e_2^k(0,.,.) = (\partial_x - \frac{p}{2\nu})e_1^{k-1}(0,.,.) = h_0^{k-1} & \text{in } \mathbb{R}\times(0,T). \end{cases}$$

Taking the Fourier transform on the equation on e_1^k in (3.1.3), we have that

$$\begin{cases} -\nu\partial_{xx}\mathfrak{F}{e_1}^k(x,k,\omega) + (i\omega+\nu k^2)\mathfrak{F}{e_1}^k(x,k,\omega) = 0 & \text{in } (0,\infty)\times\mathbb{R},\\ (\partial_x - \frac{p}{2\nu})\mathfrak{F}{e_1}^k(L,k,\omega) = \mathfrak{F}{h_L}^{k-1} & \text{in } \mathbb{R}\times(0,T). \end{cases}$$

Thus

$$\mathfrak{F}{e_1}^k(x,k,\omega) = C_1\exp(\sqrt{\frac{i\omega}{\nu}+k^2}x) + C_2\exp(-\sqrt{\frac{i\omega}{\nu}+k^2}x),$$

where $Re(\sqrt{\frac{i\omega}{\nu}+k^2}) \geq 0$.

Since $x \in (-\infty, L)$ and $\mathfrak{F}e_1^k(x,.,.) \in L^2(\mathbb{R}^2)$, we can deduce that $C_2 = 0$. Then

$$\mathfrak{F}{e_1}^k(x,k,\omega) = C_1\exp(\sqrt{\frac{i\omega}{\nu}+k^2}x).$$

Combine this equation with the boundary condition, we have that:

$$(\partial_x + \frac{p}{2\nu})\mathfrak{F}{e_1}^k(L,k,\omega) = C_1(\sqrt{\frac{i\omega}{\nu}+k^2} + \frac{p}{2\nu})\exp(\sqrt{\frac{i\omega}{\nu}+k^2}L) = \mathfrak{F}{h_L}^{k-1}.$$

Thus

$$\mathfrak{F}{e_1}^k(x,k,\omega) = (\sqrt{\frac{i\omega}{\nu} + k^2} + \frac{p}{2\nu})^{-1}(\mathfrak{F}h_L^{k-1})\exp(\sqrt{\frac{i\omega}{\nu} + k^2}(x-L)).$$

Similarly, we have that

$$\mathfrak{F}{e_2}^k(x,k,\omega) = (\sqrt{\frac{i\omega}{\nu} + k^2} - \frac{p}{2\nu})^{-1}(\mathfrak{F}{h_0}^{k-1})\exp(-\sqrt{\frac{i\omega}{\nu} + k^2}x).$$

Thus

$$(\mathfrak{F}{h_0}^k, \mathfrak{F}{h_L}^k) = \frac{2\sqrt{i\omega\nu + k^2\nu^2} - p}{2\sqrt{i\omega\nu + k^2\nu^2} + p}\exp(-\sqrt{i\omega\nu + k^2\nu^2}\frac{L}{\nu})(\mathfrak{F}{h_0}^{k-2}, \mathfrak{F}{h_L}^{k-2}).$$

Then we define the convergence factor as

$$\rho(\omega,p,k) = |\frac{2\sqrt{i\omega\nu + k^2\nu^2} - p}{2\sqrt{i\omega\nu + k^2\nu^2} + p}\exp(-\sqrt{i\omega\nu + k^2\nu^2}\frac{L}{\nu})|^2 \tag{3.1.4}$$

$$= \frac{4\sqrt{\omega^2\nu^2 + k^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}}{4\sqrt{\omega^2\nu^2 + k^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}}\exp(-\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}\frac{L}{\nu}),$$

where $\omega \in [\frac{\pi}{2T}, \frac{\pi}{\Delta t}]$, $k \in [\frac{\pi}{2Y}, \frac{\pi}{\Delta y}]$.
We need to solve the problem

$$\min_{p\in\mathbb{R}} \max_{\omega\in[\frac{\pi}{2T},\frac{\pi}{\Delta t}], k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]} \rho(\omega,p,k).$$

In fact, we only need to solve the problem

$$\min_{p\geq 0} \max_{\omega\in[\frac{\pi}{2T},\frac{\pi}{\Delta t}], k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]} \rho(\omega,p,k) = \min_{p\geq 0} ||\rho(\omega,p,k)||_\infty. \tag{3.1.5}$$

Put $\omega_1 = \frac{\pi}{2T}$, $\omega_2 = \frac{\pi}{\Delta t}$, $k_1 = \frac{\pi}{2Y}$, $k_2 = \frac{\pi}{\Delta y}$.
We have the following Theorems for the overlapping case

Theorem 3.1.1. *Suppose that L is small.*
When $\omega_2 L$ is not large (or $\omega_2 \backsim CL^{-1+\delta}, C > 0, \delta \geq 0$) and $k_2 L$ is not small (or $k_2 \backsim C'\delta^{-1+\delta'}, C' > 0, \delta' \geq 0$) we have

$$\min_{p\geq 0} \max_{\omega\in[\omega_1,\omega_2], k\in[k_1,k_2]} \rho(\omega,p,k) \backsim 1 - 2^{\frac{5}{4}} X_1^{\frac{1}{2}} \omega_2^{-\frac{1}{4}},$$

where the asymptotic expansions are due to the scale of L.

And there is only one value of p*, let say* p_**, which is the solution of this min-max problem*

$$p_* \backsim 2^{\frac{3}{4}} X_1^{\frac{1}{2}} \omega_2^{\frac{1}{4}},$$

where $X_1 = \sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}$.

When $\omega_2 L^2$ *is not small (or* $\omega_2 \backsim CL^{-2+\delta}, C, \delta \geq 0$*) and* $k_2 L$ *is not small (or* $k_2 \backsim C'\delta^{-1+\delta'}, C' > 0, \delta' \geq 0$*), we have*

$$\min_{p\geq 0} \max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]} \rho(\omega,p,k) \backsim 1 - (32\frac{LX_1}{\nu})^{\frac{1}{3}},$$

where the asymptotic expansions are due to the scale of L.

And there is only one value of p*, let say* p_**, which is the solution of this min-max problem*

$$p_* \backsim (\frac{2X_1^2\nu}{L})^{\frac{1}{3}},$$

where $X_1 = \sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}$.

Theorem 3.1.2. *For* Δx *small enough*
For $\Delta y = C_1\Delta x$*,* $\Delta t = C_2\Delta x$*,* $L = C_3\Delta x$*, we have*

$$\min_{p\geq 0} \max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]} \rho(\omega,p,k) \backsim$$

$$1 - 2^{\frac{5}{4}}(\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\pi^{-\frac{1}{4}}\Delta x^{-\frac{1}{4}}.,$$

and there is only one value of p*, let say* p_**, which is the solution of the min-max problem*

$$\min_{p\geq 0} \max_{\omega\in[\frac{\pi}{2T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]} \rho(\omega,p,k) = \max_{\omega\in[\frac{\pi}{2T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]} \rho(\omega,p_*,k),$$

and p_* *has the form*

$$p_* \backsim 2^{\frac{3}{4}}(\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\pi^{\frac{1}{4}}\Delta x^{-\frac{1}{4}}.$$

For $\Delta y = C_1\Delta x$*,* $\Delta t = C_2\Delta x^2$*,* $L = C_3\Delta x$*, we have*

$$\min_{p\geq 0} \max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]} \rho(\omega,p,k) \backsim$$

$$= 1 - 2\sqrt{2}((4\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 4(\frac{\pi}{2Y})^2\nu^2)(C_3\nu^{-1})^2)^{\frac{1}{6}}\Delta x^{\frac{1}{3}}.$$

and there is only one value of p*, let say* p_**, which is the solution of the min-max problem*

$$p_* \backsim ((4\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 4(\frac{\pi}{2Y})^2\nu^2)(C_3\nu^{-1})^{-1})^{\frac{1}{3}}\Delta x^{-\frac{1}{3}}.$$

Remark 3.1.1.

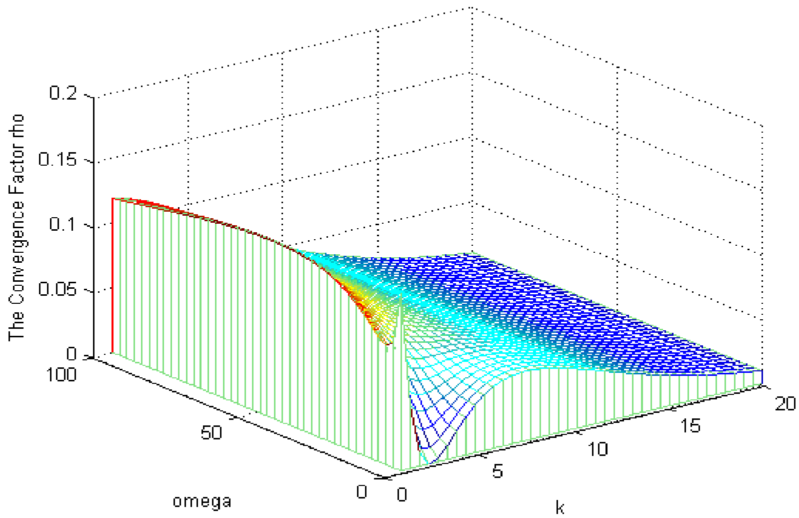


Figure 3.1.1.

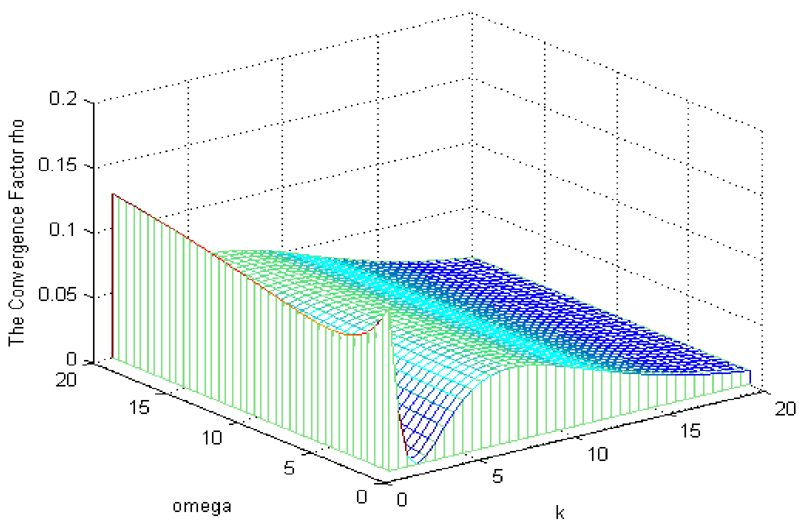


Figure 3.1.2.

Figure 3.1.1 is the graph of ρ with respect to ω for some p. In the first cases of the previous two theorems, we can prove that the solution p_* of (3.1.5) *can be obtained by equilibrating on the edge $k = k_{min}$ the two points: the* *first boundary and the maximal point (with respect to (ω_{min}, k_{min}) and the maximum point (ω_2, k_{min}) of ρ) on the graph. In the second cases $\omega_2 > \omega_{max}$, we equilibrate the two boundaries to get p_* (figure 3.1.2).*

We have the following Theorem for the nonoverlapping case

Theorem 3.1.3. *For Δx small enough, we have that*
For $\Delta y = C_1 \Delta x$, $\Delta t = C_2 \Delta x$, we have

$$\min_{p \geq 0} \max_{\omega \in [\omega_1, \omega_2], k \in [k_1, k_2]} \rho(\omega, p, k) = 1-$$

$$2\left(2\sqrt{(\frac{\pi}{2T})^2 \nu^2 + (\frac{\pi}{2Y})^4 \nu^4} + 2(\frac{\pi}{2Y})^2 \nu^2\right)^{\frac{1}{4}} \times$$

$$\times\left(\min\{\sqrt{2}\frac{C_1}{\pi\nu}, \frac{\sqrt{2\sqrt{\pi^2 C_2^{-2} \nu^2 + \pi^4 C_1^{-4} \nu^4} + 2\pi^2 C_1^{-2} \nu^2}}{\sqrt{\pi^2 C_2^{-2} \nu^2 + \pi^4 C_1^{-4} \nu^4}}\}\right)^{\frac{1}{2}} \Delta x^{\frac{1}{2}} + O(\Delta x).$$

and there is only one value of p, let say p_*, which is the solution of the min-max problem

$$p_* \backsim$$

$$2\left(2\sqrt{(\frac{\pi}{2T})^2 \nu^2 + (\frac{\pi}{2Y})^4 \nu^4} + 2(\frac{\pi}{2Y})^2 \nu^2\right)^{\frac{1}{4}} \times$$

$$\times\left(\min\{\sqrt{2}\frac{C_1}{\pi\nu}, \frac{\sqrt{2\sqrt{\pi^2 C_2^{-2} \nu^2 + \pi^4 C_1^{-4} \nu^4} + 2\pi^2 C_1^{-2} \nu^2}}{\sqrt{\pi^2 C_2^{-2} \nu^2 + \pi^4 C_1^{-4} \nu^4}}\}\right)^{-\frac{1}{2}} \Delta x^{-\frac{1}{2}}.$$

For $\Delta y = C_1 \Delta x$, $\Delta t = C_2 \Delta x^2$, we have

$$\min_{p \geq 0} \max_{\omega \in [\omega_1, \omega_2], k \in [k_1, k_2]} \rho(\omega, p, k) = 1-$$

$$2\left(2\sqrt{(\frac{\pi}{2T})^2 \nu^2 + (\frac{\pi}{2Y})^4 \nu^2} + 2(\frac{\pi}{2Y})^2 \nu^2\right)^{\frac{1}{4}} \times$$

$$\times\left(\min\{\sqrt{2}\frac{C_1}{\pi\nu}, \sqrt{\frac{2C_2}{\nu\pi}}, \frac{\sqrt{2\sqrt{\pi^2 C_2^{-2} \nu^2 + \pi^4 C_1^{-4} \nu^4} + 2\pi^2 C_1^{-2} \nu^2}}{\sqrt{\pi^2 C_2^{-2} \nu^2 + \pi^4 C_1^{-4} \nu^4}}\}\right)^{\frac{1}{2}} \Delta x^{\frac{1}{2}} + O(\Delta x),$$

and there is only one value of p, let say p_*, which is the solution of the min-max problem

$$p_* \backsim$$

$$2(2\sqrt{(\frac{\pi}{2T})^2\nu^2+(\frac{\pi}{2Y})^4\nu^4}+2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\times$$

$$\times(\min\{\sqrt{2}\frac{C_1}{\pi\nu},\sqrt{\frac{2C_2}{\nu\pi}},\frac{\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}+2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}}\})^{-\frac{1}{2}}\Delta x^{-\frac{1}{2}}.$$

Remark 3.1.2.

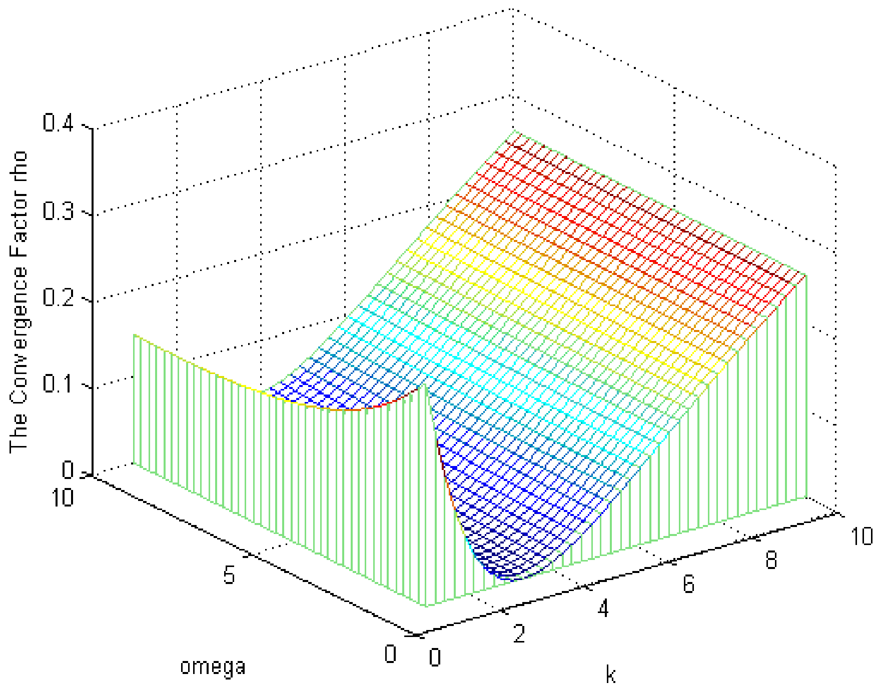


Figure 3.1.3

Figure 3.1.3 is the graph of ρ with respect to ω for some p. We equilibrate the two of the four corners to get the optimal parameters.

3.1.1 Proof of the Theorems in the Overlapping Case

Putting

$$h_L(p) = \max_{\omega \in [\frac{\pi}{T}, \frac{\pi}{\Delta t}], k \in [\frac{\pi}{2Y}, \frac{\pi}{\Delta y}]} \rho(\omega, p, k) = ||\rho(\omega, p, L)||_\infty,$$

we call that $(p^*, h_L(p^*))$ is a strictly local minimum of $h_L(p)$ iff there exists ϵ positive such that for all p in $(p^* - \epsilon, p^* + \epsilon)$, we have $h_L(p) < h_L(p^*)$.

In order to prove those theorems, we need the following lemma:

Lemma 3.1.1. *If $(p^*, h_L(p^*))$ is a strictly local minimum of $h_L(p)$, then it is the global minimum of $h_L(p)$ and p^* is the unique solution of* (3.1.5)*.*

Proof of Lemma 3.1.1

We denote $\mathcal{D}(z_0,\delta)=\{z\in\mathbb{C},|\frac{z-z_0}{z+z_0}|<\delta\}$, and $D_\delta^L=\{p|h_L(p)\leq\delta\}$.

We first prove that D_δ^L is a convex set. Let p_1 and p_2 be two elements of D_δ^L, we have that

$$|||\frac{2\sqrt{i\omega\nu+k^2\nu^2}-p_1}{2\sqrt{i\omega\nu+k^2\nu^2}+p_1}\exp(-\sqrt{i\omega\nu+k^2\nu^2}\frac{L}{\nu})|^2||_\infty\leq\delta.$$

Thus $\forall\omega\ \in\ [\omega_1,\omega_2]$, $k\in[k_1,k_2]$,

$$|\frac{2\sqrt{i\omega\nu+k^2\nu^2}-p_1}{2\sqrt{i\omega\nu+k^2\nu^2}+p_1}\exp(-\sqrt{i\omega\nu+k^2\nu^2}\frac{L}{\nu})|\leq\sqrt{\delta}.$$

Hence

$$\exp(-\sqrt{\frac{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}{2}}\frac{L}{\nu})|\frac{2\sqrt{i\omega\nu+k^2\nu^2}-p_1}{2\sqrt{i\omega\nu+k^2\nu^2}+p_1}|\leq\sqrt{\delta}.$$

Therefore

$$|\frac{2\sqrt{i\omega\nu+k^2\nu^2}-p_1}{2\sqrt{i\omega\nu+k^2\nu^2}+p_1}|\leq\sqrt{\delta}\exp(\sqrt{\frac{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}{2}}\frac{L}{\nu}).$$

This means $p_1\ \in\ \mathcal{D}(2\sqrt{i\omega\nu+k^2\nu^2},\sqrt{\delta}\exp(\sqrt{\frac{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}{2}}\frac{L}{\nu}))$.

According to Lemma 2.1 in [1], $\mathcal{D}(z_0,\delta)$ is the interior of the circle with center at $\frac{1+\delta^2}{1-\delta^2}z_0$ and radius $\frac{2\delta}{|1-\delta^2|}|z_0|$ and the exterior otherwise.

Similarly, we have also $p_2\ \in\ \mathcal{D}(2\sqrt{i\omega\nu+k^2\nu^2},\sqrt{\delta}\exp(\sqrt{\frac{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}{2}}\frac{L}{\nu}))$.

If $\sqrt{\delta}\exp(\sqrt{\frac{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}{2}}\frac{L}{\nu}))\ <\ 1$, using Lemma 2.1 in [1], we can see that $\mathcal{D}(2\sqrt{i\omega+k^2},\sqrt{\delta}\exp(\sqrt{\frac{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}{2}}\frac{L}{\nu}))$ is convex. Thus for $\theta\in[0,1]$, we have $\theta p_1+(1-\theta)p_2\in D_\delta^L$.

If $\sqrt{\delta}\exp(\frac{1}{2}\sqrt{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}\frac{L}{\nu})\ \geq\ 1$, using Lemma 2.1 in [1], we can see that for p_1, $p_2\geq 0$, $\theta\in[0,1]$, we have

$$\theta p_1+(1-\theta)p_2\in\mathcal{D}(2\sqrt{i\omega\nu+k^2\nu^2},\sqrt{\delta}\exp(\sqrt{\frac{\sqrt{\omega^2\nu^2+k^4\nu^4}+k^2\nu^2}{2}}\frac{L}{\nu})).$$

Thus for $\theta \in [0, 1]$, we have $\theta p_1 + (1-\theta)p_2 \in D^L_\delta$.

Therefore D^L_δ is convex.

Suppose that $(p^*, h_L(p^*))$ is a strictly local minimum of $h_L(p)$, we prove that it is a unique global minimum of $h_L(p)$. Suppose the contrary that there exists $(p^{**}, h_L(p^{**}))$ such that $h_L(p^*) \geq h_L(p^{**})$. Then there exists a convex neighborhood U of p^*, such that $\forall\ s \in U$, $s \neq p^*$ and $h_L(s) > h_L(p^*)$. Since $p^{**} \in D^L_{h_L(p^{**})} \subset D^L_{h_L(p^*)}$,we have that $\forall \theta \in [0, 1]$, $\theta p^* + (1-\theta)p^{**} \in D^L_{h_L(p^*)}$. For θ small enough, we have that $\theta p^{**} + (1-\theta)p^* \in U$. This is a contradiction.

Thus p^* is the unique solution of (3.1.5).

■

Proof of Theorem 3.1.1

In order to solve the problem (3.1.5), we will try to find one strictly local minimum of $\max_{\omega\in[\omega_1.\omega_2],k\in[k_1,k_2]}\rho(\omega,p,k)$, then according to Lemma 3.1.1 this local minimum is also a global minimum.

We can put $S=\frac{L}{\nu}$; then all of the asumptions: $\omega_2 L^2$ not small, $\omega_2 L^{\frac{4}{3}}$ small and $k_2 L$ not small are corrected for S in the place of L since ν is a constant.

Step 1: Finding $\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]}\rho(\omega,p,k)$ according to some particular values of p.

We consider the problem of finding the maximum for ρ. According to the Maximum Principle, the maximum values of ρ attains on the boundary of the domain. Thus, we only need to consider the maximum problem on the four edges.

Step 1.1: On the edge $\omega=\omega_1$.

$$\rho(\omega,p,k)=\frac{4\sqrt{\omega_1^2\nu^2+k^4\nu^4}+p^2-2p\sqrt{2\sqrt{\omega_1^2\nu^2+k^4\nu^4}+2k^2\nu^2}}{4\sqrt{\omega_1^2\nu^2+k^4\nu^4}+p^2+2p\sqrt{2\sqrt{\omega_1^2\nu^2+k^4\nu^4}+2k^2\nu^2}}\times$$

$$\times\exp(-\sqrt{2\sqrt{\omega_1^2\nu^2+k^4\nu^4}+2k^2\nu^2}\frac{L}{\nu}).$$

We put $X=\sqrt{2\sqrt{\omega_1^2\nu^2+k^4\nu^4}+2k^2\nu^2}$, then

$$X\in[\sqrt{2\sqrt{\omega_1^2\nu^2+k_1^4\nu^4}+2k_1^2\nu^2},\sqrt{2\sqrt{\omega_1^2\nu^2+k_2^4\nu^4}+2k_2^2\nu^2}]=[X_*,X_{**}]$$

.

Then

$$f(X)=\rho(\omega,p,k)=\frac{(X-p)^2+\frac{4\omega_1^2\nu^2}{X^2}}{(X+p)^2+\frac{4\omega_1^2\nu^2}{X^2}}\exp(-XS).$$

We will consider the behavior of f instead of ρ.

We put $a=4\omega_1^2\nu^2$, we have that

$$f(X)=\rho(\omega,p,k)=\frac{(X-p)^2+\frac{a}{X^2}}{(X+p)^2+\frac{a}{X^2}}\exp(-XS).$$

Thus

$$
\begin{aligned}
f'(X) &= \frac{\exp(-SX)}{(X^4+2pX^3+X^2p^2+a)^2}(-SX^8+2SX^6p^2-2SX^4a-SX^4p^4- \\
&\quad -2SX^2p^2a-Sa^2+4X^6p-4X^4p^3-12X^2pa) \\
&= \frac{\exp(-SX)}{(X^4+2pX^3+X^2p^2+a)^2}(-SX^8+X^6(2Sp^2+4p)- \\
&\quad -X^4(2Sa+Sp^4+4p^3)-X^2(2Sp^2a+12pa)-Sa^2).
\end{aligned}
$$

We put $Z = X^2$, then

$$Z \in [2\sqrt{\omega_1^2\nu^2+k_1^4\nu^4}+2k_1^2\nu^2, 2\sqrt{\omega_1^2\nu^2+k_2^4\nu^4}+2k_2^2\nu^2] = [Z_*, Z_{**}].$$

We denote

$$F(Z) = -SZ^4+Z^3(2Sp^2+4p)-Z^2(2Sa+Sp^4+4p^3)-Z(2Sp^2a+12pa)-Sa^2.$$

In order to consider the sign of $f'(X)$, we consider the sign of F.
Put $Z = pK$, we have that

$$
\begin{aligned}
F(Z) &= -S(pK)^4+(pK)^3(2Sp^2+4p)-(pK)^2(2Sa+Sp^4+4p^3)- \\
&\quad -(pK)(2Sp^2a+12pa)-Sa^2 \\
&= -Sp^4K^4+2Sp^5K^3+4p^4K^3-2aSp^2K^2-Sp^6K^2-4p^5K^2- \\
&\quad 2Sp^3aK-12p^2aK-Sa^2.
\end{aligned}
$$

We have that $F(Z) = 0$ is equivalent to

$$-SK^4+2SpK^3+4K^3-2a\frac{S}{p^2}K^2-Sp^2K^2-4pK^2-2\frac{S}{p}aK-\frac{12}{p^2}aK-\frac{Sa^2}{p^4} = 0.$$

Suppose that p is large, we have

$$-SK^4+(2Sp+4)K^3-Sp^2K^2-4pK^2 = 0.$$

Thus

$$-SK^2+(2Sp+4)K-Sp^2-4p = 0.$$

Hence

$$(K-p)(4-SK+Sp) = 0.$$

Therefore
$$K = p,$$
and
$$K = \frac{4}{S} + p.$$
Suppose that Sp is small, then we have two solutions of $F(Z) = 0$
$$Z_1 \backsim p^2,$$
or
$$Z_2 \backsim \frac{4p}{S} + p^2 \backsim \frac{4p}{S},$$
since $\frac{4p}{S}$ is the dominated term.

We have proved that $F(Z) = 0$ has two positive solutions. We can see that there are only two cases for the equation $F(Z) = 0$: it has two solutions or it has four solutions. If $F(Z) = 0$ has four solution, using Viète's Theorem, we can see that $F(Z) = 0$ has two positive solutions and two negative solutions. So, in any cases, we only need to consider the two positive solutions Z_1 and Z_2. Suppose that $Z_2 \in [Z_*, Z_{**}]$ or $\sqrt{\frac{p}{S}}$ is dominated by k_2, from the sign of F, we can conclude that
$$\max_{k \in [k_1, k_2]} \rho(\omega_1, p, k) = \max\{\rho(\omega_1, p, k_1), \rho(\omega_1, p, k_e)\},$$
where $X(k_e) \backsim 2\sqrt{\frac{p}{S}}$.

Step 1.2: On the edge $k = k_2$, since $k_2 S$ is not small:
$$\rho(\omega, p, k_2) \leq \exp(-\sqrt{2\sqrt{\omega^2\nu^2 + k_2^4\nu^4} + 2k_2^2\nu^2}S)$$
$$< \exp(-2k_2\nu S) < 1.$$

Therefore, the global minimum cannot be reached on this edge.

Step 1.3: On the edges $\omega = \omega_2$ and $k = k_1$ we separate the problem into two cases

Case 1 of Step 1.3: $\omega_2 S^2$ is not small

* On the edge $k = k_1$:
$$\rho(\omega, p, k_1) = \frac{4\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{4\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}} \times$$

$$\times \exp(-\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}S).$$

We put $M = \sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}$, then

$$M \in [\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}, \sqrt{2\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}] = [M_*, M_{**}].$$

Thus $2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} = M^2 - 2k_1^2\nu^2$.

We have that

$$g(M) = \rho(\omega, p, k_1) = \frac{2M^2 - 4k_1^2\nu^2 + p^2 - 2pM}{2M^2 - 4k_1^2\nu^2 + p^2 + 2pM}\exp(-MS).$$

Put $b = -4k_1^2\nu^2$, we get

$$g(M) = \rho(\omega, p, k_1) = \frac{2M^2 + b + p^2 - 2pM}{2M^2 + b + p^2 + 2pM}\exp(-MS).$$

We will consider the behavior of g instead of ρ.

We have

$$g'(M) = \frac{\exp(-SM)}{(2M^2 + 2pM + p^2 + b)}(8pM^2 - 4p^3 - 4pb - 4SM^4 - 4SM^2b - Sp^4 - 2Sp^2b - Sb^2).$$

We put

$$G(M) = -4SM^4 + M^2(8p - 4Sb) - 4p^3 - 4pb - Sp^4 - 2Sp^2b - Sb^2.$$

We have that

$$\begin{aligned} \triangle' &= (4p - 2Sb)^2 - 4S(4p^3 + 4pb + Sp^4 + 2Sp^2b + Sb^2) \\ &= 16p^2 - 16pSb + 4S^2b^2 - 16Sp^3 - 16pSb - 4S^2p^4 - 8S^2p^2b - 4S^2b^2. \end{aligned}$$

Here, we put an asumption on the largest solution of $G(M) = 0$

$$M_1 = \sqrt{\frac{4p - 2Sb + \sqrt{\Delta'}}{4S}} \backsim \sqrt{\frac{8p}{4S}} < \sqrt{2\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2} = M_{**},$$

since $\omega_2 S^2$ is not small and pS is small.

From this we can see the sign of G and the behavior of g, and we can conclude that

$$\max_{\omega\in[\omega_1,\omega_2]} \rho(\omega, p, k_1) = \{\rho(\omega_1, p, k_1), \rho(\omega_r, p, k_1)\},$$

where $M(\omega_r) \backsim \sqrt{\frac{2p}{S}}$.
* On the edge $\omega = \omega_2$, since $\omega_2 S^2$ is not small

$$\begin{aligned}\rho(\omega_2, p, k) &\leq \exp(-\sqrt{2\sqrt{\omega_2^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}S) \\ &< \exp(-\sqrt{2\omega_2\nu}S) < 1.\end{aligned}$$

Thus the global maximum cannot be attained on this edge, which means

$$\max_{\omega\in[\frac{\pi}{T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]} \rho(\omega, p, k) = \max\{\rho(\omega_1, p, k_1), \rho(\omega_1, p, k_e), \rho(\omega_r, p, k_1)\}.$$

Case 2 of Step 1.3: $\omega_2 S$ is not large
* On the edge $k = k_1$, we use exactly the same argument as in the previous case but with the following asumption

$$M_1 = \sqrt{\frac{4p - 2Sb + \sqrt{\Delta'}}{4S}} \backsim \sqrt{\frac{8p}{4S}} > \sqrt{2\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2} = M_{**}.$$

From this we can see the sign of G and the behavior of g, and we can conclude that

$$\max_{\omega\in[\omega_1,\omega_2]} \rho(\omega, p, k_1) = \{\rho(\omega_1, p, k_1), \rho(\omega_2, p, k_1)\}.$$

* On the edge $\omega = \omega_2$

$$\rho(\omega, p, k_1) = \frac{4\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{4\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}} \times$$

$$\times \exp(-\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}S).$$

Put $T = \sqrt{2\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}$.
Then

$$f(T) = \rho(\omega, p, k) = \frac{(T - p)^2 + \frac{4\omega_2^2\nu^2}{T^2}}{(T + p)^2 + \frac{4\omega_2^2\nu^2}{T^2}} \exp(-TS).$$

We put $c = 4\omega_2^2\nu^2$, we have that

$$h(T) = \rho(\omega, p, k) = \frac{(T-p)^2 + \frac{c}{T^2}}{(T+p)^2 + \frac{c}{T^2}} \exp(-TS).$$

Thus

$$\begin{aligned} h'(T) &= \frac{\exp(-ST)}{(T^4 + 2pT^3 + T^2p^2 + c)^2}(-ST^8 + 2ST^6p^2 - 2ST^4c - ST^4p^4 - 2ST^2p^2c - \\ &\quad -Sc^2 + 4T^6p - 4T^4p^3 - 12T^2pc) \\ &= \frac{\exp(-ST)}{(T^4 + 2pT^3 + T^2p^2 + c)^2}(-ST^8 + T^6(2Sp^2 + 4p) - T^4(2Sc + Lp^4 + 4p^3) - \\ &\quad -T^2(2Sp^2c + 12pc) - Sc^2). \end{aligned}$$

We put $U = T^2$, then $U \in [2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2, 2\sqrt{\omega_1^2\nu^2 + k_2^4\nu^4} + 2k_2^2\nu^2]$ $= [U_*, U_{**}]$.

We denote

$$H(U) = -SU^4 + U^3(2Sp^2 + 4p) - U^2(2Sc + Sp^4 + 4p^3) - U(2Sp^2c + 12pc) - Sc^2.$$

Put $U = pV$, we have that

$$H(U) = -Sp^4V^4 + 2Sp^5V^3 + 4p^4V^3 - 2Scp^2V^2 - Sp^6V^2 - 4p^5V^2 - 2Sp^3cV - 12p^2cV - Sc^2.$$

Hence $H(U) = 0$ means

$$-SV^4 + 2SpV^3 + 4V^3 - \frac{2Sc}{p^2}V^2 - Sp^2V^2 - 4pV^2 - 2S\frac{c}{p}V - 12\frac{c}{p^2}V - \frac{Sc^2}{p^4} = 0.$$

We put $V = \frac{W}{S}$,

$$-S(\frac{W}{S})^4 + 2Sp(\frac{W}{S})^3 + 4(\frac{W}{S})^3 - \frac{2Sc}{p^2}(\frac{W}{S})^2 - Sp^2(\frac{W}{S})^2 - 4p(\frac{W}{S})^2 - 2S\frac{c}{p}(\frac{W}{S}) - 12\frac{c}{p^2}(\frac{W}{S}) - \frac{Sc^2}{p^4} = 0.$$

Hence

$$-W^4 + 2pSW^3 + 4W^3 - 2\frac{S^2c}{p^2}W^2 - S^2p^2W^2 - 4pSW^2 - 2\frac{cS^3}{p}W - 12\frac{cS^2}{p^2}W - \frac{c^2S^4}{p^4} = 0.$$

Suppose that pS and $\frac{S^2c}{p^2}$ are small, we have that

$$-W^4 + 4W^3 = 0.$$

Hence $U \backsim \frac{4p}{S}$. We can see form this way of calculating U that U is the largest solution of $H(U) = 0$.

There are two cases for the equation $H(U) = 0$. If $H(U) = 0$ has four solutions, using the Viète Theorem, we can see that two of them are negative and the others are positive. If $H(U) = 0$ has two solutions, they are all positive. From this remark, we can deduce the sign of H and the fact that

$$\max_{k\in[k_1,k_2]} \rho(\omega_2, p, k) = \max\{\rho(\omega_2, p, k_1), \rho(\omega_2, p, k_f)\}$$

Where $T(k_f) \backsim 2\sqrt{\frac{p}{S}}$.

Thus

$$\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]} \rho(\omega_2, p, k) = \max\{\rho(\omega_1, p, k_1), \rho(\omega_1, p, k_e), \rho(\omega_2, p, k_1), \rho(\omega_2, p, k_f)\}.$$

Step 2: Using the results in Step 1, we will find a striclty minimum of $\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]} \rho(\omega, p, k)$.

In this step, we have two cases corresponding to the two cases in Step 1.

Case 1 of Step 2: $\omega_2 S^2$ is not small.

Firstly, we compute the assymptotical expansions of $\rho(\omega_1, p, k_1)$, $\rho(\omega_1, p, k_e)$ and $\rho(\omega_r, p, k_1)$.

Put $X_1 = \sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}$, we have that

$$\begin{aligned}
\rho(\omega_1, p, k_1) &= \frac{(X_1 - p)^2 + \frac{a}{X_1^2}}{(X_1 + p)^2 + \frac{a}{X_1^2}} \exp(-X_1 S) \\
&= \frac{X_1^4 - 2pX_1^3 + p^2X_1^2 + a}{X_1^4 + 2pX_1^3 + p^2X_1^2 + a} \exp(-X_1 S) \\
&= \frac{X_1^4p^{-2} - 2p^{-1}X_1^3 + X_1^2 + ap^{-2}}{X_1^4p^{-2} + 2p^{-1}X_1^3 + X_1^2 + ap^{-2}} \exp(-X_1 S) \\
&= \frac{X_1^2p^{-2} - 2p^{-1}X_1 + 1 + ap^{-2}X_1^{-2}}{X_1^2p^{-2} + 2p^{-1}X_1 + 1 + ap^{-2}X_1^{-2}} \exp(-X_1 S) \\
&= (1 - 4X_1p^{-1} + O(p^{-2})) \exp(-X_1 S) \\
&= (1 - 4X_1p^{-1} + O(p^{-2}))(1 - X_1 S + O(S^2)) \\
&\backsim 1 - 4X_1p^{-1}.
\end{aligned}$$

And

$$
\begin{aligned}
\rho(\omega_r, p, k_1) &= \frac{2M(\omega_r)^2 + b + p^2 - 2pM(\omega_r)}{2M(\omega_r)^2 + b + p^2 + 2pM(\omega_r)} \exp(-MS) \\
&= \frac{\frac{4p}{S} + b + p^2 - 2p\sqrt{\frac{2p}{S}}}{\frac{4p}{S} + b + p^2 + 2p\sqrt{\frac{2p}{S}}} \exp(-\sqrt{\frac{2p}{S}}S) \\
&= \frac{1 + \frac{bS}{4p} + \frac{pS}{4} - \frac{\sqrt{2pS}}{2}}{1 + \frac{bS}{4p} + \frac{pS}{4} + \frac{\sqrt{2pS}}{2}} \exp(-\sqrt{2pS}) \\
&\backsim 1 - 2\sqrt{2pS}.
\end{aligned}
$$

We have also

$$
\begin{aligned}
\rho(\omega_1, p, k_e) &= \frac{(X_e - p)^2 + \frac{a}{X_e^2}}{(X_e + p)^2 + \frac{a}{X_e^2}} \exp(-X_e S) \\
&= \frac{(2\sqrt{\frac{p}{S}} - p)^2 + \frac{a}{\frac{4p}{S}}}{(2\sqrt{\frac{p}{S}} + p)^2 + \frac{a}{\frac{4p}{S}}} \exp(-2\sqrt{\frac{p}{S}}S) \\
&= \frac{\frac{4p}{S} - 4\sqrt{\frac{p^3}{S}} + p^2 + \frac{aS}{4p}}{\frac{4p}{S} + 4\sqrt{\frac{p^3}{S}} + p^2 + \frac{aS}{4p}} \exp(-2\sqrt{\frac{p}{S}}S) \\
&= \frac{1 - \sqrt{pS} + \frac{pS}{4} + \frac{aS^2}{16p^2}}{1 + \sqrt{pS} + \frac{pS}{4} + \frac{aS^2}{16p^2}} \exp(-2\sqrt{pS}) \\
&\backsim 1 - 4\sqrt{pS}.
\end{aligned}
$$

Thus

$$||\rho(\omega, p, k)||_\infty = \max\{\rho(\omega_1, p, k_1), \rho(\omega_r, p, k_1)\}. \tag{3.1.6}$$

Secondly, We find the solution p_* of the following equation approximately

$$\rho(\omega_1, p, k_1) = \rho(\omega_r, p, k_1).$$

We solve

$$4X_1 p_*^{-1} = 2\sqrt{2p_* S}.$$

We have

$$2X_1^2 p_*^{-2} = p_* S.$$

Thus

$$p_* \backsim (\frac{2X_1^2}{S})^{\frac{1}{3}}.$$

Next, we will check the conditions that we have used to archive p_*.
Firstly, we can easily see that p_*S is small and $\sqrt{\frac{p}{S}} < k_2$.
Secondly, we will verify the condition

$$M_1 = \sqrt{\frac{4p_* - 2Sb + \sqrt{\Delta'}}{4S}} \backsim \sqrt{\frac{2p_*}{S}} < \sqrt{2\sqrt{\omega_2^2 + k_1^4} + 2k_1^2}.$$

Which is equivalent to

$$\frac{(\frac{2X_1^2}{S})^{\frac{1}{3}}}{S} < 2\sqrt{\omega_2^2 + k_1^4} + 2k_1^2,$$

or

$$2^{\frac{1}{3}} X_1^{\frac{2}{3}} S^{-\frac{4}{3}} < 2\sqrt{\omega_2^2 + k_1^4} + 2k_1^2.$$

This is right because $\omega_2 S^{\frac{4}{3}}$ is not small.
Finally, we prove that this p_* is a strictly local minimum of ρ.
We have that

$$\frac{\partial}{\partial p}\rho(\omega_1, p, k_1) = \frac{4\exp(-L\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2})\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{(4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2})^2} \times$$

$$\times (p^2 - 4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4}).$$

For p closed to p_*, $\frac{\partial}{\partial p}\rho(\omega_1, p, k_1) > 0$.

$$\frac{\partial}{\partial p}\rho(\omega_r, p, k_1) = \frac{4\exp(-L\sqrt{2\sqrt{\omega_r^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2})\sqrt{2\sqrt{\omega_r^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{(4\sqrt{\omega_r^2\nu^2 + k_1^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega_r^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2})^2} \times$$

$$\times \frac{\partial \omega_r}{\partial p}(p^2 - 4\sqrt{\omega_r^2\nu^2 + k_1^4\nu^4}).$$

Since $\frac{\partial \omega_r}{\partial p} \backsim \nu^{-1}(2X_1^2)^{-\frac{4}{3}}$, then for p closed to p_*, $\frac{\partial}{\partial p}\rho(\omega_r, p, k_1) < 0$.
For p closed to p_*, $p > p_*$, we have that $\max\{\rho(\omega_1, p, k_1), \rho(\omega_r, p, k_1)\} =$

$\rho(\omega_1,p,k_1) > \rho(\omega_1,p_*,k_1) = \rho(\omega_r,p_*,k_1)$. And, for p closed to p_*, $p > p_*$, we have that $\max\{\rho(\omega_1,p,k_1),\rho(\omega_r,p,k_1)\} = \rho(\omega_r,p,k_1) > \rho(\omega_1,p_*,k_1) = \rho(\omega_r,p_*,k_1)$

Hence the value p_*, where $\rho(\omega_1,p_*,k_1) = \rho(\omega_r,p_*,k_1)$ is a strictly local minimum of $\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]}\rho(\omega,p,k)$.

And

$$\begin{aligned}\max_{\omega\in[\frac{\pi}{T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]}\rho(\omega,p,k) &\backsim 1-\frac{4X_1}{p}\\ &= 1-\frac{4X_1}{(\frac{2X_1^2}{S})^{\frac{1}{3}}}\\ &= 1-(32SX_1)^{\frac{1}{3}}.\end{aligned}$$

Hence, when $\omega_2 S^2$ is not small and $k_2 S$ is not small, we have

$$p_* \backsim (\frac{2X_1^2}{S})^{\frac{1}{3}},$$

and

$$\min_{p\in\mathbb{R}}\max_{\omega\in[\frac{\pi}{T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]}\rho(\omega,p,k) \backsim 1-(32SX_1)^{\frac{1}{3}},$$

where $X_1 = \sqrt{2\sqrt{\omega_1^2+k_1^4}+2k_1^2}$.

Case 2 of Step 2: $\omega_2 S$ is not large.

Similar as in Case 1 of Step 2, we have that

$$\begin{aligned}\rho(\omega_2,p,k_f) &\backsim 1-4\sqrt{pS},\\ \rho(\omega_2,p,k_1) &\backsim 1-\frac{\sqrt{2}p}{\sqrt{\omega_2}}-\sqrt{2\omega_2}S,\\ \rho(\omega_1,p,k_1) &\backsim 1-4X_1p^{-1},\\ \rho(\omega_1,p,k_e) &\backsim 1-4\sqrt{pS}.\end{aligned}$$

Using the same argument as in the previous case, we have that our solution p_* is the solution of the following system of asymptotic equations

$$\rho(\omega_2,p,k_1) = \rho(\omega_1,p,k_1).$$

We have

$$p_* \backsim 2^{\frac{1}{8}}\sqrt{X_1}\omega_2^{\frac{1}{4}}.$$

We can verify that this p_* with the conditions that we have assumed.

Hence, when $\omega_2 S^2$ small, and $k_2 S$ not small, we have

$$p_* \backsim 2^{\frac{1}{8}} \sqrt{X_1} \omega_2^{\frac{1}{4}}.$$

and

$$\min_{p \in \mathbb{R}} \max_{\omega \in [\frac{\pi}{T}, \frac{\pi}{\Delta t}], k \in [\frac{\pi}{2Y}, \frac{\pi}{\Delta y}]} \rho(\omega, p, k) \backsim 1 - 2^{\frac{5}{4}} X_1^{\frac{1}{2}} \omega_2^{-\frac{1}{4}},$$

where $X_1 = \sqrt{2\sqrt{\omega_1^2 \nu^2 + k_1^4 \nu^4} + 2k_1^2 \nu^2}$.

■

Proof of Theorem 3.1.2
Case 1: $\Delta y = C_1\Delta x$, $\Delta t = C_2\Delta x$, $L = C_3\Delta x$.
Similar as in the previous theorem, we get

$$p_* \backsim 2^{\frac{3}{4}}(\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\pi^{\frac{1}{4}}\Delta x^{-\frac{1}{4}}.$$

Hence

$$\min_{p\geq 0}\max_{\omega\in[\frac{\pi}{2T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]}\rho(\omega,p,k) =$$

$$1 - 2^{\frac{5}{4}}(\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\pi^{-\frac{1}{4}}\Delta x^{-\frac{1}{4}}.,$$

Case 2: $\Delta y = C_1\Delta x$, $\Delta t = C_2\Delta x^2$, $L = C_3\Delta x$.
We put $S = (C_3\nu^{-1})\Delta x$.
Similar as in the proof of Theorem 3.1.1, we find that

$$p_* \backsim ((4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 4k_1^2\nu^2)(C_3\nu^{-1})^{-1})^{\frac{1}{3}}\Delta x^{-\frac{1}{3}},$$

and

$$\min_{p\geq 0}\max_{\omega\in[\frac{\pi}{2T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]}\rho(\omega,p,k) =$$

$$= 1 - 2\sqrt{2}((4\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 4(\frac{\pi}{2Y})^2\nu^2)(C_3\nu^{-1})^2)^{\frac{1}{6}}\Delta x^{\frac{1}{3}} + O(\Delta x^{\frac{2}{3}}).$$

■

3.1.2 Proof of the Theorems in the Nonoverlapping Case

Proof of Theorem 3.1.3
Step 1: Similar as in the proof of Theorems 3.1.1 and 3.1.2, we will first consider the problem of finding $\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]}\rho$ and we only consider this problem on the four edges.
Step 1.1:
On the edge $\omega = \omega_1$.

$$\rho(\omega,p,k) = \frac{4\sqrt{\omega_1^2\nu^2 + k^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega_1^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}}{4\sqrt{\omega_1^2\nu^2 + k^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega_1^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}}.$$

We put $X = \sqrt{2\sqrt{\omega_1^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}$, then

$$X \in [\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}, \sqrt{2\sqrt{\omega_1^2\nu^2 + k_2^4\nu^4} + 2k_2^2\nu^2}] = [X_*, X_{**}].$$

Then

$$f(X) = \rho(\omega, p, k) = \frac{(X-p)^2 + \frac{4\omega_1^2\nu^2}{X^2}}{(X+p)^2 + \frac{4\omega_1^2\nu^2}{X^2}}.$$

We will consider the behavior of f instead of ρ.

$$f'(X) = \frac{4X^2 p(X^4 - p^2X^2 - 12\omega_1^2\nu^2)}{(X^4 + 2pX^3 + X^2p^2 + 4\omega_1^2\nu^2)^2}.$$

Since the equation $X^4 - p^2X^2 - 12\omega_1^2\nu^2 = 0$ has one positive solution which is $\sqrt{\frac{p^2+\sqrt{p^4+48\omega_1^2\nu^2}}{2}}$, then in both cases $X_{**} \geq \sqrt{\frac{p^2+\sqrt{p^4+48\omega_1^2\nu^2}}{2}}$ or $X_{**} < \sqrt{\frac{p^2+\sqrt{p^4+48\omega_1^2\nu^2}}{2}}$, we always have that $\max_{X\in[X_*,X_{**}]} f(X) = \max\{f(X_*), f(X_{**})\}$.

Thus

$$\max_{k\in[k_1,k_2]} \rho(\omega_1, p, k) = \max\{\rho(\omega_1, p, k_1), \rho(\omega_1, p, k_2)\}.$$

Step 1.2: On the edge $\omega = \omega_2$. Similarly, we also have that

$$\max_{k\in[k_1,k_2]} \rho(\omega_2, p, k) = \max\{\rho(\omega_2, p, k_1), \rho(\omega_2, p, k_2)\}.$$

Step 1.3: On the edge $k = k_1$. We have that

$$\rho(\omega, p, k_1) = \frac{4\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{4\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}.$$

We put $M = \sqrt{2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}$, then

$$M \in [\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}, \sqrt{2\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}] = [M_*, M_{**}].$$

Thus $2\sqrt{\omega^2\nu^2 + k_1^4\nu^4} = M^2 - 2k_1^2\nu^2$.

We have that

$$g(M) = \rho(\omega, p, k_1) = \frac{2M^2 - 4k_1^2\nu^2 + p^2 - 2pM}{2M^2 - 4k_1^2\nu^2 + p^2 + 2pM}.$$

$$g'(M) = \frac{2M^2 - p^2 + 4k_1^2\nu^2}{(2M^2 - 4k_1^2\nu^2 + p^2 + 2pM)^2}.$$

In this case, we have also that

$$\max_{\omega\in[\omega_1,\omega_2]} \rho(\omega, p, k_1) = \max\{\rho(\omega_1, p, k_1), \rho(\omega_2, p, k_1)\}.$$

Step 1.4: On the edge $k = k_2$, we have also that

$$\max_{\omega\in[\omega_1,\omega_2]} \rho(\omega, p, k_2) = \max\{\rho(\omega_1, p, k_2), \rho(\omega_2, p, k_2)\}.$$

Combining those four cases, we have that

$$\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]} \rho(\omega, p, k) = \max\{\rho(\omega_1, p, k_1), \rho(\omega_2, p, k_1), \rho(\omega_1, p, k_2), \rho(\omega_2, p, k_2)\}.$$

Suppose that $p = C_p\Delta^{-\gamma_p}$, where $0 < \gamma_p \leq \frac{1}{2}$, we will consider the asymptotic behavior of the four points $\rho(\omega_1, p, k_1), \rho(\omega_2, p, k_1), \rho(\omega_1, p, k_2), \rho(\omega_2, p, k_2)$.

Step 2: We equilibrate the equations to get the solutions.

We will consider the following two cases:

Case 1: $\Delta y = C_1\Delta x$ and $\Delta t = C_2\Delta x^2$

We have that

$$\begin{aligned}
\rho(\omega_1, p, k_1) &= \frac{4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}} \\
&= \frac{4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + C_p^2\Delta x^{-2\gamma_p} - 2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + C_p^2\Delta x^{-2\gamma_p} + 2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}} \\
&= [4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4}C_p^{-2}\Delta x^{2\gamma_p} + 1 - 2C_p^{-1}\Delta x^{\gamma_p}\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}] \times \\
&\quad \times[4\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4}C_p^{-2}\Delta x^{2\gamma_p} + 1 + 2C_p^{-1}\Delta x^{\gamma_p}\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}]^{-1} \\
&= 1 - 4C_p^{-1}\Delta x^{\gamma_p}\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2} + O(\Delta x^{2\gamma_p}),
\end{aligned}$$

$$
\begin{aligned}
\rho(\omega_1, p, k_2) &= \frac{4\sqrt{\omega_1^2\nu^2 + k_2^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega_1^2\nu^2 + k_2^4\nu^4} + 2k_2^2\nu^2}}{4\sqrt{\omega_1^2\nu^2 + k_2^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega_1^2\nu^2 + k_2^4\nu^4} + 2k_2^2\nu^2}} \\
&= [4\sqrt{\omega_1^2\nu^2 + \pi^4 C_1^{-4}\Delta x^{-4}\nu^4} + C_p^2\Delta x^{-2\gamma_p} - \\
&\quad -2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\omega_1^2\nu^2 + \pi^4 C_1^{-4}\Delta x^{-4}\nu^4} + 2\pi^2 C_1^{-2}\Delta x^{-2}}] \times \\
&\quad \times [4\sqrt{\omega_1^2\nu^2 + \pi^4 C_1^{-4}\Delta x^{-4}\nu^4} + C_p^2\Delta x^{-2\gamma_p} + \\
&\quad +2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\omega_1^2\nu^2 + \pi^4 C_1^{-4}\Delta x^{-4}}\nu^4 + 2\pi^2 C_1^{-2}\Delta x^{-2}\nu^2}]^{-1} \\
&= 1 - 2\frac{C_p C_1}{\nu\pi}\Delta x^{1-\gamma_p} + O(\Delta x^{2-2\gamma_p}),
\end{aligned}
$$

$$
\begin{aligned}
\rho(\omega_2, p, k_1) &= \frac{4\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + p^2 - 2p\sqrt{2\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}}{4\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + p^2 + 2p\sqrt{2\sqrt{\omega_2^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}} \\
&= [4\sqrt{\pi^2 C_2^{-2}\Delta x^{-4}\nu^2 + k_1^4\nu^4} + C_p^2\Delta x^{-2\gamma_p} - \\
&\quad -2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\pi^2 C_2^{-2}\Delta x^{-4}\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}] \times \\
&\quad \times [4\sqrt{\pi^2 C_2^{-2}\Delta x^{-4}\nu^2 + k_1^4\nu^4} + C_p^2\Delta x^{-2\gamma_p} + \\
&\quad +2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\pi^2 C_2^{-2}\Delta x^{-4}\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2}]^{-1} \\
&= 1 - \sqrt{\frac{2C_2}{\nu\pi}}C_p\Delta x^{1-\gamma_p} + O(\Delta x^{2-2\gamma_p}),
\end{aligned}
$$

and

$$
\begin{aligned}
\rho(\omega_2,p,k_2) &= \frac{4\sqrt{\omega_2^2\nu^2+k_2^4\nu^4}+p^2-2p\sqrt{2\sqrt{\omega_2^2\nu^2+k_2^4\nu^4}+2k_2^2\nu^2}}{4\sqrt{\omega_2^2\nu^2+k_2^4\nu^4}+p^2+2p\sqrt{2\sqrt{\omega_2^2\nu^2+k_2^4\nu^4}+2k_2^2\nu^2}} \\
&= [4\sqrt{\pi^2C_2^{-2}\Delta x^{-4}\nu^2+\pi^4C_1^{-4}\Delta x^{-4}\nu^4}+C_p^2\Delta x^{-2\gamma_p}- \\
&\quad -2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\pi^2C_2^{-2}\Delta x^{-4}\nu^2+\pi^4C_1^{-4}\nu^4\Delta x^{-4}}+2\pi^2C_1^{-2}\nu^2\Delta x^{-2}}]\times \\
&\quad \times[4\sqrt{\pi^2C_2^{-2}\Delta x^{-4}\nu^2+\pi^4C_1^{-4}\Delta x^{-4}\nu^4}+C_p^2\Delta x^{-2\gamma_p}+ \\
&\quad +2C_p\Delta x^{-\gamma_p}\sqrt{2\sqrt{\pi^2C_2^{-2}\Delta x^{-4}\nu^2+\pi^4C_1^{-4}\nu^4\Delta x^{-4}}+2\pi^2C_1^{-2}\nu^2\Delta x^{-2}}]^{-1} \\
&= 1-\frac{C_p\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}+2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}}\Delta x^{1-\gamma_p}+O(\Delta x^{2-2\gamma_p}).
\end{aligned}
$$

Using Lemma 3.1.1 and the same argument as in the previous section, we will try to find a strictly local minimum p_* by equilibrating these asymptotic expansions. First, we can equilibrate the orders γ_p and $1-\gamma_p$ to get $\gamma_p=\frac{1}{2}$, then we can equilibrate the coefficients

$$
4C_p^{-1}\sqrt{2\sqrt{\omega_1^2\nu^2+k_1^4\nu^4}+2k_1^2\nu^2}=C_p\min\{2\frac{C_1}{\pi\nu},\sqrt{2\frac{C_2}{\pi\nu}},
$$

$$
\frac{\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}+2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}}\}.
$$

Thus

$$
C_p=2(2\sqrt{(\frac{\pi}{2T})^2\nu^2+(\frac{\pi}{2Y})^4\nu^4}+2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\times
$$

$$
\times(\min\{2\frac{C_1}{\pi\nu},\sqrt{\frac{2C_2}{\nu\pi}},\frac{\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}+2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2+\pi^4C_1^{-4}\nu^4}}\})^{-\frac{1}{2}}.
$$

Using the same argument as in the previous section, we can see that this p is the solution of our problem.

Then

$$
\min_{p\in\mathbb{R}}\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]}\rho(\omega,p,k)=
$$

$$= 2(2\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\times$$

$$\times(\min\{2\frac{C_1}{\pi\nu}, \sqrt{\frac{2C_2}{\nu\pi}}, \frac{\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4} + 2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4}}\})^{\frac{1}{2}} + O(\Delta x).$$

Case 2: $\Delta y = C_1\Delta x$ and $\Delta t = C_2\Delta x$, similar as in the previous cases, we have the following results

$$\begin{aligned}
\rho(\omega_1, p, k_1) &= 1 - 4C_p^{-1}\Delta x^{\gamma_p}\sqrt{2\sqrt{\omega_1^2 + k_1^4} + 2k_1^2} + O(\Delta x^{2\gamma_p}),\\
\rho(\omega_1, p, k_2) &= 1 - 2\frac{C_pC_1}{\nu\pi}\Delta x^{1-\gamma_p} + O(\Delta x^{2-2\gamma_p}),\\
\rho(\omega_2, p, k_1) &= 1 - \sqrt{\frac{2C_2}{\nu\pi}}C_p\Delta x^{\frac{1}{2}-\gamma_p} + O(\Delta x^{2-2\gamma_p}),\\
\rho(\omega_2, p, k_2) &= 1 - \frac{C_p\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4} + 2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4}}\Delta x^{1-\gamma_p} + O(\Delta x^{2-2\gamma_p}).
\end{aligned}$$

Using Lemma 3.1.1 and the same argument as in the previous section, we will try to find a strictly local minimum p_* by equilibrating these asymptotic expansions. First, we can equilibrate the orders γ_p and $1-\gamma_p$ to get $\gamma_p = \frac{1}{2}$, then we can equilibrate the coefficients

$$4C_p^{-1}\sqrt{2\sqrt{\omega_1^2\nu^2 + k_1^4\nu^4} + 2k_1^2\nu^2} = C_p\min\{2\frac{C_1}{\pi\nu}, \frac{\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4} + 2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4}}\}.$$

Thus

$$C_p = 2(2\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}}\times$$

$$\times(\min\{2\frac{C_1}{\pi\nu}, \frac{\sqrt{2\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4} + 2\pi^2C_1^{-2}\nu^2}}{\sqrt{\pi^2C_2^{-2}\nu^2 + \pi^4C_1^{-4}\nu^4}}\})^{-\frac{1}{2}}.$$

Using the same argument as in the previous section, we can see that this p is the solution of our problem.

Then

$$\min_{p\in\mathbb{R}}\max_{\omega\in[\omega_1,\omega_2],k\in[k_1,k_2]}\rho(\omega, p, k) =$$

$$= 2(2\sqrt{(\frac{\pi}{2T})^2\nu^2 + (\frac{\pi}{2Y})^4\nu^4} + 2(\frac{\pi}{2Y})^2\nu^2)^{\frac{1}{4}} \times$$

$$\times(\min\{2\frac{C_1}{\pi\nu}, \frac{\sqrt{2\sqrt{\pi^2 C_2^{-2}\nu^2 + \pi^4 C_1^{-4}\nu^4} + 2\pi^2 C_1^{-2}\nu^2}}{\sqrt{\pi^2 C_2^{-2}\nu^2 + \pi^4 C_1^{-4}\nu^4}}\})^{\frac{1}{2}}\Delta x^{\frac{1}{2}} + O(\Delta x).$$

■

3.2 Optimized Schwarz Waveform Relaxation Methods For The Two Dimensional Heat Equation With Ventcell Transmission Condition

In this section, we are interested in the following heat equation:

$$\begin{cases} \mathfrak{L}u = \partial_t u - \nu\partial_{xx}u - \nu\partial_{yy}u = f & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.2.1}$$

We consider the following algorithm

$$\begin{cases} \mathfrak{L}u_1^k = f & \text{in } (-\infty, L) \times \mathbb{R} \times (0,T), \\ u_1^k(x,y,0) = u_0(x,y) & \text{in } (-\infty, L) \times \mathbb{R}, \\ \mathfrak{B}_1 u_1^k(L,.,.) = \mathfrak{B}_1 u_2^{k-1}(L,.,.) & \text{in } \mathbb{R} \times (0,T), \end{cases} \tag{3.2.2}$$

$$\begin{cases} \mathfrak{L}u_2^k = f & \text{in } (0,\infty) \times \mathbb{R} \times (0,T), \\ u_2^k(x,y,0) = u_0(x,y) & \text{in } (0,\infty) \times \mathbb{R}, \\ \mathfrak{B}_2 u_2^k(0,.,.) = \mathfrak{B}_2 u_1^{k-1}(0,.,.) & \text{in } \mathbb{R} \times (0,T), \end{cases}$$

where

$$\mathfrak{B}_1 = \partial_x + \frac{1}{2\nu}\mathfrak{S},$$

$$\mathfrak{B}_2 = \partial_x - \frac{1}{2\nu}\mathfrak{S},$$

$$\mathfrak{S} = p + 4q\nu(\partial_t - \nu\Delta_y).$$

Using the same Fourier technique as in the previous section, we can define the convergence factor as

$$\begin{aligned} \rho(\omega,p,k) &= |\frac{2\sqrt{i\omega\nu + k^2\nu^2} - p - q(i\omega\nu + k^2\nu^2)}{2\sqrt{i\omega\nu + k^2\nu^2} + p + q(i\omega\nu + k^2\nu^2)} \exp(-\sqrt{i\omega\nu + k^2\nu^2}\frac{L}{\nu})|^2 \\ &= \frac{\rho_1}{\rho_2}\rho_3, \end{aligned} \tag{3.2.3}$$

where

$$\rho_1 = 4\sqrt{\omega^2\nu^2 + k^4\nu^4} + p^2 + q^2(\omega^2\nu^2 + k^4\nu^4) + 2pqk^2\nu^2$$
$$-2(p + qk^2\nu^2)\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} - 2q\omega\nu\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} - 2k^2\nu^2},$$
$$\rho_2 = 4\sqrt{\omega^2\nu^2 + k^4\nu^4} + p^2 + q^2(\omega^2\nu^2 + k^4\nu^4) + 2pqk^2\nu^2$$
$$+2(p + qk^2\nu^2)\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} + 2q\omega\nu\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} - 2k^2\nu^2},$$
$$\rho_3 = \exp(-\sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}\frac{L}{\nu}).$$

We need to consider the following min-max problem:

$$\min_{p,q\in\mathbb{R}} \max_{k\in[k_{min},k_{max}],\omega\in[\omega_{min},\omega_{max}]} \rho(\omega, k, p, q).$$

Which is equivalent to the following min-max problem:

$$\min_{p,q\geq 0} \max_{k\in[k_{min},k_{max}],\omega\in[\omega_{min},\omega_{max}]} \rho(\omega, k, p, q).$$

We have the following Theorems

Theorem 3.2.1. *When $k_{max}L$ is not small and $\omega_{max}L$ is not small and not large we have*

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega, k, p, q) \backsim 1 - 2^{\frac{9}{5}} X_{min}^{\frac{1}{5}} (L\nu^{-1})^{\frac{1}{5}},$$

where the asymptotic expansions are due to the scale of L.

And there is only one value of (p, q), let say (p_*, q_*), which is the solution of the equation

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega, k, p, q) = \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega, k, p_*, q_*),$$

and p_* has the form

$$q_* \backsim 2^{\frac{8}{5}} (L\nu^{-1})^{\frac{3}{5}} X_1^{-\frac{2}{5}},$$

and

$$p_* \backsim 2^{-\frac{1}{5}} (L\nu^{-1})^{-\frac{1}{5}} X_{min}^{\frac{4}{5}}.$$

where $X_{min} = \sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}$.
When $k_{max}L$ *is not small and* $\omega_{max}L^2$ *is not small we have*

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) \backsim 1 - 4X_{min}^{\frac{1}{5}}(L\nu^{-1})^{\frac{1}{5}},$$

where the asymptotic expansions are due to the scale of L.

And there is only one value of (p,q), *let say* (p_*,q_*), *which is the solution of the equation*

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) = \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p_*,q_*),$$

and p_* *has the form*

$$q_* \backsim 2(L\nu^{-1})^{\frac{3}{5}}X_{min}^{-\frac{2}{5}}.$$

Therefore

$$p_* \backsim (L\nu^{-1})^{-\frac{1}{5}}X_{min}^{\frac{4}{5}}.$$

where $X_{min} = \sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}$.

Using this theorem, we can have the following results for the overlapping case

Theorem 3.2.2. *If* $\Delta y = C_1\Delta x$, $\Delta t = C_2\Delta x$ *and* $L = C_3\Delta x$ *we have*

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) \backsim 1 - 2^{\frac{9}{5}}X_{min}^{\frac{1}{5}}(L\nu^{-1})^{\frac{1}{5}}.$$

And there is only one value of (p,q), *let say* (p_*,q_*), *which is the solution of this min-max problem*

$$q_* \backsim 2^{\frac{8}{5}}(L\nu^{-1})^{\frac{3}{5}}X_1^{-\frac{2}{5}},$$

and

$$p_* \backsim 2^{-\frac{1}{5}}(L\nu^{-1})^{-\frac{1}{5}}X_{min}^{\frac{4}{5}}.$$

where $X_{min} = \sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}$.
If $\Delta y = C_1\Delta x$, $\Delta t = C_2\Delta x^2$ *and* $L = C_3\Delta x$ *we have*

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) \backsim 1 - 4X_{min}^{\frac{1}{5}}(L\nu^{-1})^{\frac{1}{5}}.$$

And there is only one value of (p,q)*, let say* (p_*,q_*)*, which is the solution of this min-max problem*

$$q_* \backsim 2(L\nu^{-1})^{\frac{3}{5}} X_{min}^{-\frac{2}{5}}.$$

Therefore

$$p_* \backsim (L\nu^{-1})^{-\frac{1}{5}} X_{min}^{\frac{4}{5}}.$$

where $X_{min} = \sqrt{2\sqrt{\omega_{min}^2 \nu^2 + k_{min}^4 \nu^4} + 2k_{min}^2 \nu^2}.$

Remark 3.2.1.

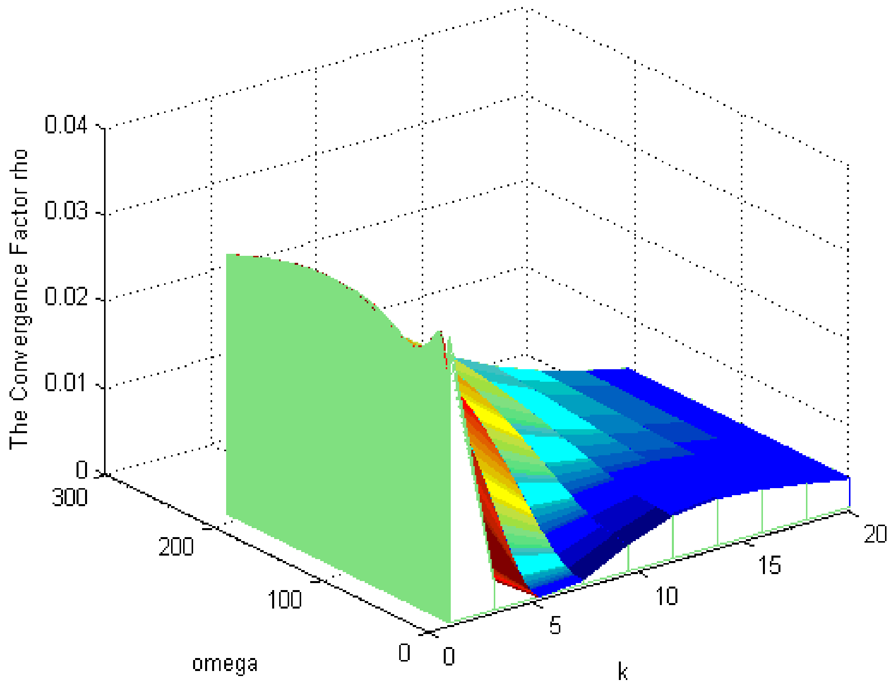


Figure 3.2.1.

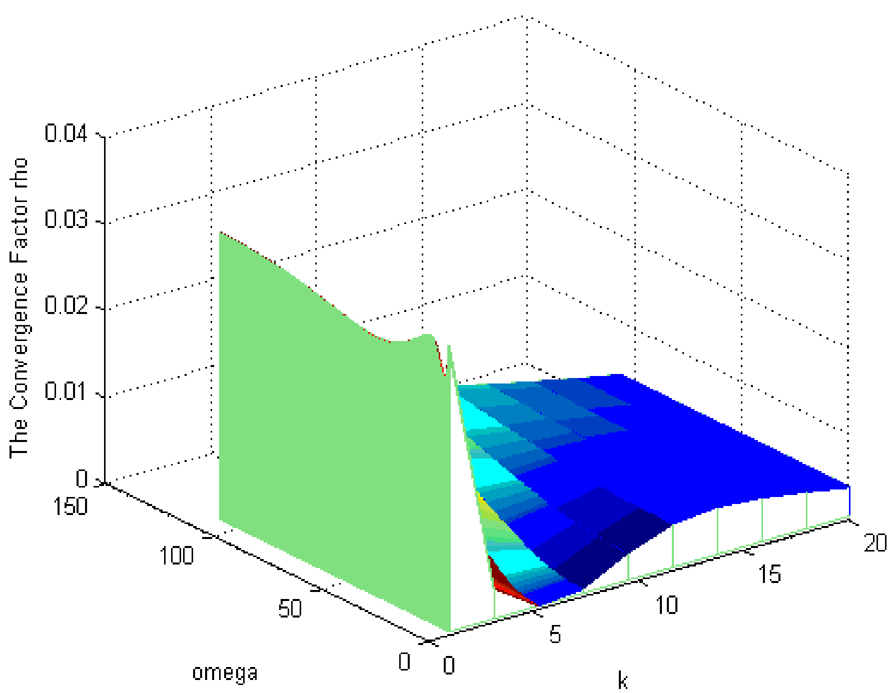


Figure 3.2.2.

Figures 3.2.1 and 3.2.2 are the graphs of ρ *with respect to* ω *for some* (p, q).

In the first cases of the previous two theorems, we can prove that the solution (p_*, q_*) of (3.2.3) can be obtained by equilibrating on the edge $k = k_{min}$ the three points: the first boundary and the two maximal points (with respect to (ω_{min}, k_{min}) and the maximum point (ω_2, k_{min}), (ω_4, k_{min}) of ρ) on the graph. In the second cases $\omega_2 > \omega_{max}$, we equilibrate the two boundaries and (ω_2, k_{min}) to get (p_*, q_*).

We have the following Theorem for the nonoverlapping case

Theorem 3.2.3. *For Δx small enough*
For $\Delta t = C_1 \Delta x^2$, $\Delta y = C_2 \Delta x$, we have

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) \backsim 1 - 2^{\frac{7}{4}} X_{min}^{-\frac{3}{4}} (\frac{\pi\nu}{C_2})^{-\frac{1}{8}} A^{\frac{1}{4}} \Delta x^{\frac{1}{4}},$$

where

$$A = \min\{\sqrt{2}, \frac{k_{max}^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k_{max}^4\nu^4} + 2k_{max}^2\nu^2} + \omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k_{max}^4\nu^4} - 2k_{max}^2\nu^2}}{(\omega_{max}^2\nu^2 + k_{max}^4\nu^4)\sqrt{\omega_{max}\nu}}\},$$

and $X_{min} = \sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}$. There is only one value of (p,q), let say (p_*, q_*), which is the solution of this min-max problem

$$p_* = 2^{\frac{1}{4}} X_{min}^{\frac{3}{4}} (\omega_{max}\nu)^{\frac{1}{8}} A^{-\frac{1}{4}} = 2^{\frac{1}{4}} X_{min}^{\frac{3}{4}} (\frac{\pi\nu}{C_1})^{\frac{1}{8}} A^{-\frac{1}{4}} \Delta x^{-\frac{1}{4}},$$

and

$$q_* = 2^{\frac{1}{4}} X_{min}^{-\frac{1}{4}} (\omega_{max})^{-\frac{3}{8}} A^{\frac{3}{4}} = 2^{\frac{1}{4}} X_{min}^{-\frac{1}{4}} (\frac{\pi\nu}{C_1})^{-\frac{3}{8}} A^{\frac{3}{4}} \Delta x^{\frac{3}{4}}.$$

For $\Delta t = C_1 \Delta x$, $\Delta y = C_2 \Delta x$, we have

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) \backsim 1 - 4X_{min}^{\frac{1}{4}} (\frac{\pi\nu}{C_2})^{-\frac{1}{4}} \Delta x^{\frac{1}{4}}.$$

There is only one value of (p,q), let say (p_*, q_*), which is the solution of this min-max problem

$$p_* = X_{min}^{\frac{3}{4}} k_{max}^{\frac{1}{4}} \nu^{\frac{1}{4}} = 1 - X_{min}^{\frac{3}{4}} (\frac{\pi\nu}{C_2})^{\frac{1}{4}} \Delta x^{-\frac{1}{4}}.$$

Therefore

$$q_* = 2X_{min}^{-\frac{1}{4}} k_{max}^{-\frac{3}{4}} \nu^{-\frac{3}{4}} = 2X_{min}^{-\frac{1}{4}} (\frac{\pi\nu}{C_2})^{-\frac{3}{4}} \Delta x^{\frac{3}{4}}.$$

Remark 3.2.2.

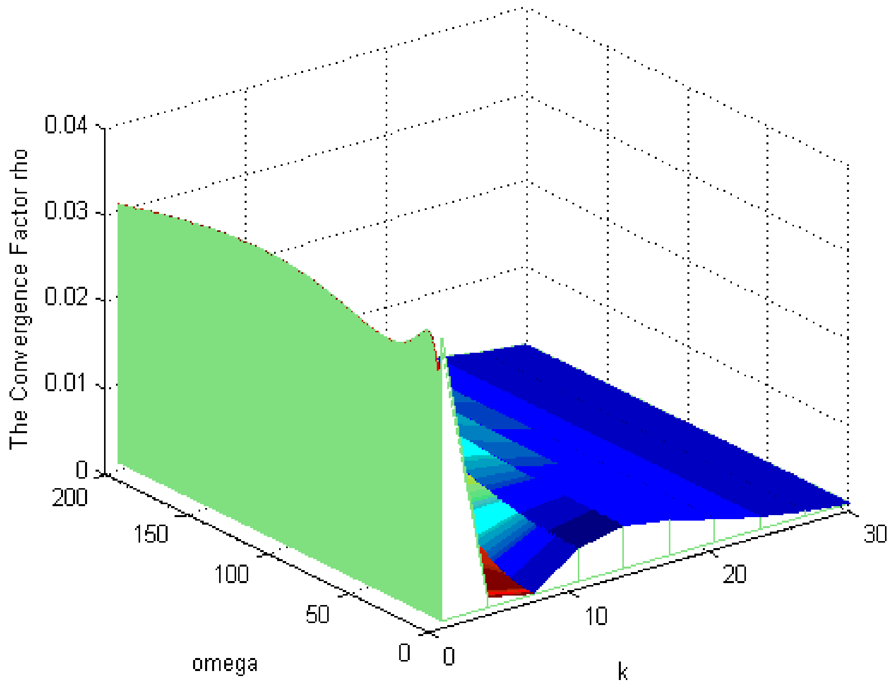


Figure 3.2.3.

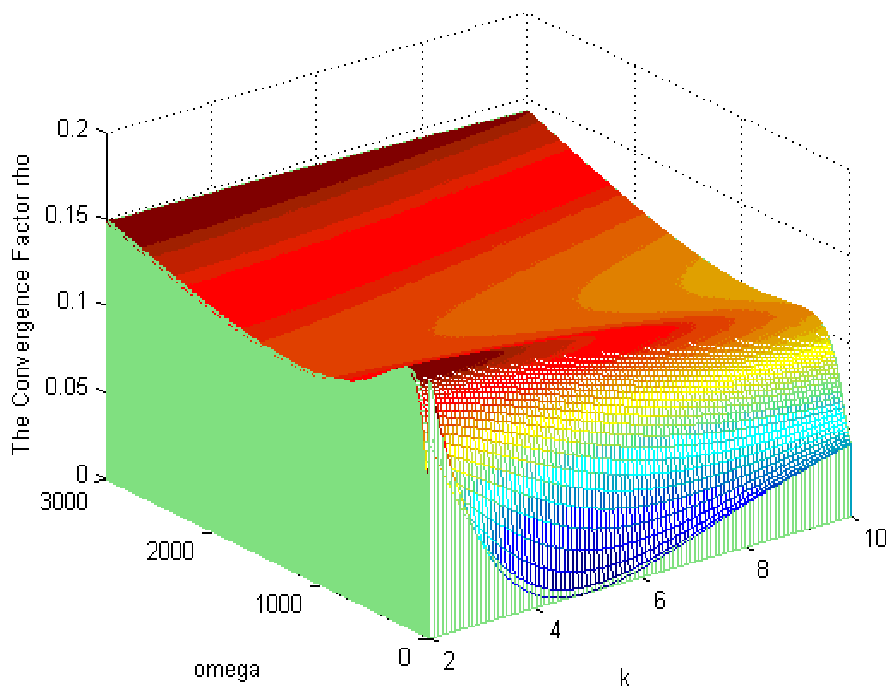


Figure 3.2.4.

Figures 3.2.3 and 3.2.4 are the graphs of ρ with respect to ω for some (p, q)

for the nonoverlapping case. Similarly, the solution (p_*, q_*) *of* (3.2.3) *can be obtained by equilibrating on the edge* $k = k_{min}$ *the three points: the first boundary and the two maximal points (figure 3.2.3) or the two boundaries the maximal point (figure 3.2.4).*

3.2.1 Proof of the Theorems in the Overlapping Case

Putting

$$h_L(p,q) = \max_{\omega\in[\frac{\pi}{2T},\frac{\pi}{\Delta t}],k\in[\frac{\pi}{2Y},\frac{\pi}{\Delta y}]} \rho(\omega,k,p,q) = ||\rho(\omega,k,p,q)||_\infty,$$

we call that $h_L(p^*,q^*)$ is a strictly local minimum of $h_L(p,q)$ iff there exists ϵ positive such that for all (p,q) in $(p^*-\epsilon,p^*+\epsilon)\times(q^*-\epsilon,q^*+\epsilon)$, we have $h_L(p,q) < h_L(p^*,q^*)$.

In order to prove those theorems, we need the following lemma, whose proof is similar with the previous ones.

Lemma 3.2.1. *If $h_L(p^*,q^*)$ is a strictly local minimum of $h_L(p,q)$, then it is the global minimum of $h_L(p,q)$.*

Proof of Theorem 3.2.1

Case 1: $k_{max}L$ **is not small and** $\omega_{max}L^2$ **is not small**

Step 1 of Case 1: Similar as in the previous sections, we will consider the max problem:

$$\max_{k\in[k_{min},k_{max}],\omega\in[\omega_{min},\omega_{max}]} \rho(\omega,k,p,q),$$

where $p = C_pS^{-\gamma_p}$, $q = C_qS^{-\gamma_q}$, and $|\gamma_p| + |\gamma_q| < 1$, $\gamma_p \geq 0 \geq \gamma_q$, $2\gamma_p + \gamma_q < 1$, $|\gamma_q| > |\gamma_p|$. According to the maximum principle, we can see that the maximum can only be archived on the four edges of the domain. We denote by S the value $\frac{L}{\nu}$.

Step 1.1 of Case 1: On the edge $k = k_{max}$, we have that

$$\rho \leq \rho_3 = \exp(-\sqrt{2\sqrt{\omega^2\nu^2 + k_{max}^4\nu^4} + 2k_{max}^2\nu^2}\frac{L}{\nu})$$

$$\leq \exp(-2k_{max}\nu\frac{L}{\nu}) = \exp(-2k_{max}L) < C_1 < 1,$$

where C_1 is a constant, since $k_{max}L$ is not small.

Step 1.2 of Case 1: On the edge $\omega = \omega_{max}$, we have that

$$\rho \leq \rho_3 = \exp(-\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}\frac{L}{\nu})$$

$$\leq \exp(-\sqrt{2\omega_{max}\nu}\frac{L}{\nu}) < C_2 < 1,$$

where C_2 is a constant, since $\omega_{max}L^2$ is not small.

Step 1.3 of Case 1: On the edge $\omega = \omega_{min}$, we put $X = \sqrt{2\sqrt{\omega_{min}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}$ and $a = \omega_{min}\nu$. Thus $X \in [X_{min}, X_{max}] = [\sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2},$ $\sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{max}^4\nu^4} + 2k_{max}^2\nu^2}]$. We have also

$$\sqrt{\omega_{min}^2\nu^2 + k^4\nu^4} = \frac{a^2}{X^2} + \frac{X^2}{4},$$

and

$$k^2\nu^2 = \frac{X^2}{4} - \frac{a^2}{X^2}.$$

Put these values into the formula of ρ, we have that

$$\rho = \frac{\rho_1}{\rho_2}\rho_3 =: f_1(X),$$

where

$$\begin{aligned}
\rho_1 &= \frac{4a^2}{X^2} + X^2 + p^2 + q^2(\frac{a^2}{X^2} + \frac{X^2}{4})^2 + 2pq(\frac{X^2}{4} - \frac{a^2}{X^2}) - \\
&\quad -2(p + q(\frac{X^2}{4} - \frac{a^2}{X^2}))X - 2qa\frac{2a}{X}, \\
\rho_2 &= \frac{4a^2}{X^2} + X^2 + p^2 + q^2(\frac{a^2}{X^2} + \frac{X^2}{4})^2 + 2pq(\frac{X^2}{4} - \frac{a^2}{X^2}) + \\
&\quad +2(p + q(\frac{X^2}{4} + \frac{a^2}{X^2}))X - 2qa\frac{2a}{X}, \\
\rho_3 &= \exp(-XS).
\end{aligned}$$

We consider now the behavior of f_1. We have

$$f_1'(X) = g_1(X)h_1(X),$$

where

$$\begin{aligned}
g_1(X) &= -\exp(-XS)(16X^6 + 64a^2X^2 + 16p^2X^4 + q^2X^8 + 8q^2X^4a^2 + 16q^2a^4 + \\
&\quad +32pX^5 + 8qX^7 + 32qa^2X^3 + 8pqX^6 - 32pqX^2a^2)^{-2},
\end{aligned}$$

and

$$\begin{aligned}
h_1(X) &= -4096X^4Sa^4pq + 64X^{10}Sq^3pa^2 - 2048X^8Spqa^2 + \\
&\quad +1536X^4Sq^2a^4p^2 - 1024X^{10}p + 256X^{12}q - 16q^3X^{14} + 1024p^3X^8 + \\
&\quad 256X^{12}S + 3072q^2a^4pX^4 - 3584q^2X^8pa^2 - 7168qa^2X^6p^2 - \\
&\quad -128X^{10}q^2a^2 + 512X^6Sq^2a^4 - 256X^{12}Spq + 2048X^6Sa^2p^2 + \\
&\quad +2048X^2Sa^6q^2 + 96X^{12}Sp^2q^2 + 16X^{12}Sq^4a^2 + 96X^8Sq^4a^4 + \\
&\quad +16X^{14}Sq^3p + 256X^4Sq^4a^6 + 256X^{10}Sp^3q + 256Sq^4a^8 + \\
&\quad +2048X^8qa^2 + 12288a^2X^6p + 4096a^4X^4q - 64q^2X^{12}p + \\
&\quad +64q^3X^{10}a^2 + 1280q^3a^4X^6 + 3072q^3a^6X^2 + 256qX^{10}p^2 + \\
&\quad +2048X^8Sa^2 - 512X^{10}Sp^2 - 32X^{14}Sq^2 + 4096X^4Sa^4 + \\
&\quad +X^{16}Sq^4 - 256X^6Sq^3a^4p - 1024X^2Sq^3a^6p + 256X^8Sp^4 - \\
&\quad -256X^8Sp^2q^2a^2 - 1024X^6Sp^3qa^2.
\end{aligned}$$

We can see that

$$\max X \in [X_{min}, X_{max}] f_1(X) = \max\{f_1(X_{min}), f_1(X_{max}), f_1(X_i)\},$$

where $f'(X_i) = 0$.

Thus, we will look for the solutions of the equation $h_1(X) = 0$ in the interval $[X_{min}, X_{max}]$. Suppose that $X \backsim C_X S^{-\gamma_X}$ where $C_X, \gamma_X > 0$, we will solve the equation $h_1(X) = 0$ asymptotically. We have that

$$\begin{aligned}
h_1(X) &= pq(-4096X^4Sa^4 - 2048X^8Sa^2 - 256X^{12}S) \\
&\quad +q^3p(64X^{10}Sa^2 + 16X^{14}S - 256X^6Sa^4 - 1024X^2Sa^6) \\
&\quad +q^2p^2(1536X^4Sa^4 + 96X^{12}S - 256X^8Sa^2) \\
&\quad +p(-1024X^{10} + 12288a^2X^6) \\
&\quad +q(256X^{12} + 2048X^8a^2 + 4096a^4X^4) \\
&\quad +q^3(-16X^{14} + 64X^{10}a^2 + 1280a^4X^6 + 3072a^6X^2) \\
&\quad +p^3(1024X^8) \\
&\quad +q^2p(3072a^4X^4 - 3584X^8a^2 - 64X^{12}) \\
&\quad +qp^2(-7168a^2X^6 + 256X^{10}) \\
&\quad +q^2(-128X^{10}Sa^2 + 512X^6Sa^4 + 2048X^2Sa^6 - 32X^{14}S) \\
&\quad +p^2(2048X^6Sa^2 - 512X^{10}S) \\
&\quad +q^4(16X^{12}Sa^2 + 96X^8Sa^4 + 256X^4Sa^6 + 256Sa^8 + X^{16}S) \\
&\quad +p^3q(256X^{10}S - 1024X^6Sa^2) \\
&\quad +p^4(256X^8S) \\
&\quad +4096X^4Sa^4 + 256X^{12}S + 2048X^8Sa^2.
\end{aligned}$$

We can consider the equation $h_1(X) = 0$ as an equation of L. Here, we care about the highest order of $\frac{1}{S}$ in the equation, thus from the formula above, we can have the following equation

$$\begin{aligned}
0 &= -pq256X^{12}S + q^3p16X^{14}S + p^2q^296X^{12}S - p1024X^{10} + q256X^{12} - \\
&\quad -q^316X^{14} + p^31024X^8 - q^2p64X^{12} + qp^2256X^{10} - q^232X^{14}S - \\
&\quad -p^2512X^{10}S + q^4X^{16}S + p^3q256X^{10}S + p^4256X^8S + 256X^{12}S,
\end{aligned}$$

or we have

$$\begin{aligned}
0 &= q^4SX^{16} + X^{14}(q^3pS16 - q^316 - q^232S) + X^{12}(-pq256S + p^2q^296S + \\
&\quad +q256 - q^2p64 + 256S) + X^{10}(-p1024 + qp^2256 - p^2512S + p^3q256S) + \\
&\quad +X^8(p^31024 + p^4256S)
\end{aligned}$$

Since we care only about the highest order of $\frac{1}{S}$ in the equation, we can reduce the equation into

$$q^4SX^{16} - X^{14}q^3 16 + X^{12} 256q - X^{10}p1024 + X^8p^3 1024 = 0.$$

Using the same argument as in the previous section, we can see that this equation has four solutions $X_1 \backsim p$, $X_2 \backsim 2p^{\frac{1}{2}}q^{-\frac{1}{2}}$, $X_3 \backsim 4q^{-\frac{1}{2}}S^{-\frac{1}{2}}$ and $X_4 \backsim 4q^{-1}$. And we can see that $X_1, X_2, X_3, X_4 \in [X_{min}, X_{max}]$. Thus

$$\max_{X \in [X_{min}, X_{max}]} f_1(X) = \max\{f_1(X_{min}), f_1(X_{max}), f_1(X_1), f_1(X_2), f_1(X_3), f_1(X_4)\}.$$

* For $f_1(X_{min})$, we have that

$$\begin{aligned}
\rho_1 &\backsim p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) - 2pX_{min}, \\
\rho_2 &\backsim p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) + 2pX_{min}, \\
\rho_3 &\backsim 1 - X_{min}S.
\end{aligned}$$

Thus

$$\begin{aligned}
f_1(X_{min}) &= \frac{\rho_1}{\rho_2}\rho_3 \backsim \frac{p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) - 2pX_{min}}{p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) + 2pX_{min}}(1 - X_{min}S). \\
&\backsim 1 - \frac{4X_{min}}{p}.
\end{aligned}$$

* For $f(X_1)$, we have that

$$\begin{aligned}
\rho_1 &\backsim 4a^2p^{-2} + p^2 + p^2 + q^2(a^2p^{-2} + \frac{p^2}{4})^2 + 2pq(\frac{p^2}{4} - a^2p^{-2}) \\
&\quad -2(p + q(\frac{p^2}{4} - a^2p^{-2}))p - 4a^2qp^{-1} \\
&\backsim \frac{q^2p^4}{16} \\
\rho_2 &\backsim 4a^2p^{-2} + p^2 + p^2 + q^2(a^2p^{-2} + \frac{p^2}{4})^2 + 2pq(\frac{p^2}{4} - a^2p^{-2}) + \\
&\quad +2(p + q(\frac{p^2}{4} - a^2p^{-2}))p + 4a^2qp^{-1} \\
&\backsim 4p^2,
\end{aligned}$$

and

$$\rho_3 \backsim 1 - pS.$$

Thus

$$f_1(X_1) \backsim \frac{q^2p^4}{64p^2} = \frac{p^2q^2}{64}.$$

* For $f_1(X_2)$, we have that

$$\begin{aligned}
\rho_1 \quad \backsim \quad & 4a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2} + (2p^{\frac{1}{2}}q^{-\frac{1}{2}})^2 + p^2 + q^2(a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2} + pq^{-1})^2 + 2pq(pq^{-1} - \\
& -a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2}) - 2(p + q(pq^{-1} - a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2}))2p^{\frac{1}{2}}q^{-\frac{1}{2}} - 4a^2q\frac{1}{2}p^{-\frac{1}{2}}q^{\frac{1}{2}} \\
\backsim \quad & 4pq^{-1} + p^2 + q^2p^2q^{-2} + 2p^2 - 8p^{\frac{3}{2}}q^{-\frac{1}{2}} \\
\backsim \quad & 4pq^{-1} + 4p^2 - 8p^{\frac{3}{2}}q^{-\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\rho_2 \quad \backsim \quad & 4a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2} + (2p^{\frac{1}{2}}q^{-\frac{1}{2}})^2 + p^2 + q^2(a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2} + pq^{-1})^2 + 2pq(pq^{-1} - \\
& -a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2}) + 2(p + q(pq^{-1} + a^2(2p^{\frac{1}{2}}q^{-\frac{1}{2}})^{-2}))2p^{\frac{1}{2}}q^{-\frac{1}{2}} + 4a^2q\frac{1}{2}p^{-\frac{1}{2}}q^{\frac{1}{2}} \\
\backsim \quad & 4pq^{-1} + p^2 + q^2p^2q^{-2} + 2p^2 - 8p^{\frac{3}{2}}q^{-\frac{1}{2}} \\
\backsim \quad & 4pq^{-1} + 4p^2 + 8p^{\frac{3}{2}}q^{-\frac{1}{2}},
\end{aligned}$$

and

$$\rho_3 \backsim 1 - 2p^{\frac{1}{2}}q^{-\frac{1}{2}}S.$$

Thus

$$\begin{aligned}
\rho \quad = \quad & \frac{\rho_1}{\rho_2}\rho_3 \backsim \frac{4pq^{-1} + 4p^2 - 8p^{\frac{3}{2}}q^{-\frac{1}{2}}}{4pq^{-1} + 4p^2 + 8p^{\frac{3}{2}}q^{-\frac{1}{2}}}(1 - 2p^{\frac{1}{2}}q^{-\frac{1}{2}}S) \\
\backsim \quad & 1 - 4p^{\frac{1}{2}}q^{\frac{1}{2}}.
\end{aligned}$$

* For $f_1(X_3)$, we have that

$$\begin{aligned}
\rho_1 \quad \backsim \quad & \frac{4a^2}{16}qS + 16q^{-1}S^{-1} + p^2 + q^2(\frac{a^2}{16}qS + 4q^{-1}S^{-1})^2 + 2pq(4q^{-1}S^{-1} - \frac{a^2}{16}qS) \\
& -2(p + q(4q^{-1}S^{-1} - \frac{a^2}{16}qS))4q^{-\frac{1}{2}}S^{-\frac{1}{2}} - \frac{4a^2q}{4}q^{\frac{1}{2}}S^{\frac{1}{2}} \\
\backsim \quad & 16q^{-1}S^{-1} + p^2 + 16S^{-2} + 8pS^{-1} - 8pq^{-\frac{1}{2}}S^{-\frac{1}{2}} - 32q^{-\frac{1}{2}}S^{-\frac{3}{2}},
\end{aligned}$$

$$\begin{aligned}
\rho_2 &\backsim \frac{4a^2}{16}qS + 16q^{-1}S^{-1} + p^2 + q^2(\frac{a^2}{16}qS + 4q^{-1}S^{-1})^2 + 2pq(4q^{-1}S^{-1} - \frac{a^2}{16}qS) \\
&+2(p + q(4q^{-1}S^{-1} - \frac{a^2}{16}qS))4q^{-\frac{1}{2}}S^{-\frac{1}{2}} + \frac{4a^2q}{4}q^{\frac{1}{2}}S^{\frac{1}{2}} \\
&\backsim 16q^{-1}S^{-1} + p^2 + 16S^{-2} + 8pS^{-1} + 8pq^{-\frac{1}{2}}S^{-\frac{1}{2}} + 32q^{-\frac{1}{2}}S^{-\frac{3}{2}},
\end{aligned}$$

and

$$\rho_3 \backsim 1 - 4q^{-\frac{1}{2}}S^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}
f_1(X_3) &\backsim \frac{16q^{-1}S^{-1} + p^2 + 16S^{-2} + 8pS^{-1} - 8pq^{-\frac{1}{2}}S^{-\frac{1}{2}} - 32q^{-\frac{1}{2}}S^{-\frac{3}{2}}}{16q^{-1}S^{-1} + p^2 + 16S^{-2} + 8pS^{-1} + 8pq^{-\frac{1}{2}}S^{-\frac{1}{2}} + 32q^{-\frac{1}{2}}S^{-\frac{3}{2}}}(1 - 4q^{-\frac{1}{2}}S^{\frac{1}{2}}) \\
&\backsim \frac{16S^{-2} - 32q^{-\frac{1}{2}}S^{-\frac{3}{2}}}{16S^{-2} + 32q^{-\frac{1}{2}}S^{-\frac{3}{2}}}(1 - 4q^{-\frac{1}{2}}S^{\frac{1}{2}}) \\
&\backsim 1 - 8q^{-\frac{1}{2}}S^{\frac{1}{2}}.
\end{aligned}$$

* For $f_1(X_4)$, we have that

$$\begin{aligned}
\rho_1 &\backsim \frac{4a^2}{16q^{-2}} + 16q^{-2} + p^2 + q^2(\frac{a^2}{16q^{-2}} + 4q^{-2})^2 + 2pq(4q^{-2} - \frac{a^2}{16q^{-2}}) \\
&-2(p + q(4q^{-2} - \frac{a^2}{16q^{-2}}))4q^{-1} - \frac{4a^2q}{4q^{-1}} \\
&\backsim 16q^{-2} + 16q^{-2} + p^2 + 8pq^{-1} - 8pq^{-1} - 32q^{-2} \backsim p^2,
\end{aligned}$$

$$\begin{aligned}
\rho_2 &\backsim \frac{4a^2}{16q^{-2}} + 16q^{-2} + p^2 + q^2(\frac{a^2}{16q^{-2}} + 4q^{-2})^2 + 2pq(4q^{-2} - \frac{a^2}{16q^{-2}}) \\
&+2(p + q(4q^{-2} + \frac{a^2}{16q^{-2}}))4q^{-1} - \frac{4a^2q}{4q^{-1}} \\
&\backsim 16q^{-2} + 16q^{-2} + p^2 + 8pq^{-1} + 8pq^{-1} + 32q^{-2} \backsim 64q^{-2},
\end{aligned}$$

and

$$\rho_3 \backsim 1 - 4q^{-1}S.$$

Thus

$$f_1(X_4) \backsim \frac{p^2q^2}{64}.$$

* For $f_1(X_{max})$: we do not need to consider this case, because in this case $k = k_{max}$ and we have considered it.

Hence

$$\max_{X\in[X_{min},X_{max}]} f_1(X) = \max\{f_1(X_{max}), 1-\frac{4X_1}{p}, 1-4p^{\frac{1}{2}}q^{\frac{1}{2}}, 1-8S^{\frac{1}{2}}q^{-\frac{1}{2}}\}.$$

Step 1.4 of Case 1: On the edge $k = k_{min}$, we put $K = \sqrt{2\sqrt{\omega^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}$, and $a = k_{min}^2\nu^2$, then

$$K \in [K_{min}, K_{max}] = [\sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}, \sqrt{2\sqrt{\omega_{max}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}].$$

Thus

$$\sqrt{\omega^2\nu^2 + k_{min}^4\nu^4} = \frac{K^2}{2} - a,$$

and

$$\begin{aligned} \omega\nu\sqrt{2\sqrt{\omega^2\nu^2 + a^2} - 2a} &= \frac{2\omega^2\nu^2}{\sqrt{2\sqrt{\omega^2\nu^2 + a^2} + 2a}} = \frac{2\omega^2\nu^2}{K} = \frac{2[(\frac{K^2}{2} - a)^2 - a^2]}{K} \\ &= \frac{2}{K}[\frac{K^4}{4} - K^2 a] = \frac{K^3}{2} - 2Ka. \end{aligned}$$

Putting these formulas into ρ, we have that

$$f_2(K) = \rho(\omega, k, p, q) = \frac{\rho_1}{\rho_2}\rho_3,$$

where

$$\rho_1 = 4(\frac{K^2}{2} - a) + p^2 + q^2(\frac{K^2}{2} - a)^2 + 2pqa - 2(p + qa)K - 2q(\frac{K^3}{2} - 2Ka),$$

$$\rho_2 = 4(\frac{K^2}{2} - a) + p^2 + q^2(\frac{K^2}{2} - a)^2 + 2pqa + 2(p + qa)K + 2q(\frac{K^3}{2} - 2Ka),$$

and

$$\rho_3 = \exp(-KS).$$

The max problem turns into the following problem

$$\max_{K\in[K_{min},K_{max}]} f_2(K).$$

We can see that

$$\max_{K\in[K_{min},K_{max}]} f_2(K) = \max\{f_2(K_{min}), f_2(K_{max}), f_2(K_i)\},$$

where K_i's are the points where $f_2'(K_i) = 0$.

We have that

$$f_2'(K) = g_2(K)h_2(K),$$

where

$$g_2(K) = \exp(-KS)(8K^2 - 16a + 4p^2 + q^2K^4 - 4q^2K^2a + 4q^2a^2 + 8Kp - 8Kpa + 4qK^3 + 8pqa)^{-2},$$

and

$$\begin{aligned} h_2(K) \;=\;& 16Sq^4a^4 - 256SK^2a - 64Sq^3K^2a^2p + 16Sq^3K^4pa - \\ & -32Sp^2q^2K^2a - 256K^2qa - 48q^2K^4p + 16q^3K^4a + 32q^3K^2a^2 + \\ & +64p^2qa + 96qK^2p^2 - 64pq^2a^2 + 64p^3 - 128K^2p + 64qK^4 - 8q^3K^6 - \\ & -256pa + 256qa^2 - 64q^3a^3 + 256SK^2pqa + Sq^4K^8 - 128Sap^2 - \\ & -128Sq^2a^3 + 16Sp^4 + 64SK^4 + 256Sa^2 + 256q^2K^2ap - 32SK^4q^2a + \\ & +128SK^2q^2a^2 - 256Spqa^2 + 96Sp^2q^2a^2 + 64Sp^3qa - 8Sq^4K^6a + \\ & +24Sq^4K^4a^2 - 32Sq^4K^2a^3 + 8Sp^2q^2K^4 + 64Sq^3a^3p - 64SK^4pq. \end{aligned}$$

We have to solve the equation $h_2(K) = 0$. Using the same argument as above, we reduce this equation into

$$K^8q^4S - 8q^3K^6 + 64qK^4 - 128pK^2 + 64p^3 = 0.$$

This equation has four solutions $K_1 \backsim 2\sqrt{2}q^{-\frac{1}{2}}S^{-\frac{1}{2}}$, $K_2 \backsim 2\sqrt{2}q^{-1}$, $K_3 \backsim \sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}$, $K_4 \backsim \frac{\sqrt{2}}{2}p$. We can see that $K_1, K_2, K_3, K_4 \in [K_{min}, K_{max}]$. Thus

$$\max_{K\in[K_{min},K_{max}]} f_2(K) = \max\{f_2(K_{min}), f_2(K_{max}), f_2(K_1), f_2(K_2), f_2(K_3), f_2(K_4)\}.$$

We do not need to consider K_{min} and K_{max} in this case because we have already consider the max problem on the edges $\omega = \omega_{min}$, $\omega = \omega_{max}$.

* For $f_2(K_1)$, we have that

$$\begin{aligned} \rho_1 \;\backsim\;& 16q^{-1}S^{-1} + p^2 + q^2(4q^{-1}S^{-1} - a)^2 - 2(p+qa)2\sqrt{2}q^{-\frac{1}{2}}S^{-\frac{1}{2}} \\ & -2q(8\sqrt{2}q^{-\frac{3}{2}}S^{-\frac{3}{2}} - 4\sqrt{2}aq^{-\frac{1}{2}}S^{-\frac{1}{2}}) + 2pqa \\ \backsim\;& 16q^{-1}S^{-1} + p^2 + 16S^{-2} - 4\sqrt{2}pq^{-\frac{1}{2}}S^{-\frac{1}{2}} - 16\sqrt{2}q^{-\frac{1}{2}}S^{-\frac{3}{2}} \\ \backsim\;& 16S^{-2} - 16\sqrt{2}q^{-\frac{1}{2}}S^{-\frac{3}{2}}, \end{aligned}$$

$$
\begin{aligned}
\rho_2 &\backsim 16q^{-1}S^{-1} + p^2 + q^2(4q^{-1}S^{-1} - a)^2 + 2(p + qa)2\sqrt{2}q^{-\frac{1}{2}}S^{-\frac{1}{2}} \\
&\quad +2q(8\sqrt{2}q^{-\frac{3}{2}}S^{-\frac{3}{2}} - 4\sqrt{2}aq^{-\frac{1}{2}}S^{-\frac{1}{2}}) + 2pqa \\
&\backsim 16q^{-1}S^{-1} + p^2 + 16S^{-2} + 4\sqrt{2}pq^{-\frac{1}{2}}S^{-\frac{1}{2}} + 16\sqrt{2}q^{-\frac{1}{2}}S^{-\frac{3}{2}} \\
&\backsim 16S^{-2} + 16\sqrt{2}q^{-\frac{1}{2}}S^{-\frac{3}{2}},
\end{aligned}
$$

and

$$\rho_3 \backsim 1 - 2\sqrt{2}q^{-\frac{1}{2}}S^{\frac{1}{2}}.$$

Thus

$$f_2(K_1) \backsim 1 - 4\sqrt{2}q^{-\frac{1}{2}}S^{\frac{1}{2}}.$$

* For $f_2(K_2)$, we have that

$$
\begin{aligned}
\rho_1 &\backsim 16q^{-2} - 4a + p^2 + q^2(4q^{-2} - a)^2 - 2(p + qa)2\sqrt{2}q^{-1} - \\
&\quad -2q(8\sqrt{2}q^{-3} - 4\sqrt{2}aq^{-1}) + 2pqa \\
&\backsim 32q^{-2} + p^2 - 4\sqrt{2}pq^{-1} - 16\sqrt{2}q^{-2} \\
&\backsim (32 - 16\sqrt{2})q^{-2},
\end{aligned}
$$

$$
\begin{aligned}
\rho_2 &\backsim 16q^{-2} - 4a + p^2 + q^2(4q^{-2} - a)^2 + 2(p + qa)2\sqrt{2}q^{-1} + \\
&\quad +2q(8\sqrt{2}q^{-3} - 4\sqrt{2}aq^{-1}) + 2pqa \\
&\backsim 32q^{-2} + p^2 + 4\sqrt{2}pq^{-1} + 16\sqrt{2}q^{-2} \\
&\backsim (32 + 16\sqrt{2})q^{-2},
\end{aligned}
$$

and

$$\rho_3 \backsim 1 - 2\sqrt{2}q^{-1}S.$$

Thus

$$f_2(K_2) \backsim \frac{2 - \sqrt{2}}{2 + \sqrt{2}}.$$

* For $f_2(K_3)$, we have that

$$
\begin{aligned}
\rho_1 &\backsim 4pq^{-1} - 4a + p^2 + q^2(pq^{-1} - a)^2 - 2(p + qa)\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}} - \\
&\quad -2q(\sqrt{2}p^{\frac{3}{2}}q^{-\frac{3}{2}} - 2a\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}) + 2pqa \\
&\backsim 4pq^{-1} + p^2 + p^2 - 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} - 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} \\
&\backsim 4pq^{-1} + 2p^2 - 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}},
\end{aligned}
$$

$$\begin{aligned}\rho_2 &\backsim 4pq^{-1} - 4a + p^2 + q^2(pq^{-1} - a)^2 + 2(p + qa)\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}} + \\ &\quad +2q(\sqrt{2}p^{\frac{3}{2}}q^{-\frac{3}{2}} - 2a\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}) + 2pqa \\ &\backsim 4pq^{-1} + p^2 + p^2 + 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} + 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} \\ &\backsim pq^{-1} + 2p^2 + 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}},\end{aligned}$$

and

$$\rho_3 \backsim 1 - \sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}S.$$

Thus

$$\begin{aligned}f_2(K_3) &\backsim \frac{4pq^{-1} + 2p^2 - 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}}}{4pq^{-1} + 2p^2 + 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}}}(1 - \sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}S) \\ &\backsim 1 - 2\sqrt{2}p^{\frac{1}{2}}q^{\frac{1}{2}}.\end{aligned}$$

* For $f_2(K_4)$, we have that

$$\begin{aligned}\rho_1 &\backsim p^2 - 4a + p^2 + q^2(\frac{p^2}{4} - a)^2 - 2(p + qa)\frac{\sqrt{2}p}{2} - 2q(\frac{p^3}{4\sqrt{2}} - \sqrt{2}pa) + 2pqa \\ &\backsim p^2 + p^2 + \frac{q^2p^4}{16} - \sqrt{2}p^2 - \frac{qp^3}{\sqrt{8}} \\ &\backsim (2 - \sqrt{2})p^2,\end{aligned}$$

$$\begin{aligned}\rho_2 &\backsim p^2 - 4a + p^2 + q^2(\frac{p^2}{4} - a)^2 + 2(p + qa)\frac{\sqrt{2}p}{2} + 2q(\frac{p^3}{4\sqrt{2}} - \sqrt{2}pa) + 2pqa \\ &\backsim p^2 + p^2 + \frac{q^2p^4}{16} + \sqrt{2}p^2 + \frac{qp^3}{\sqrt{8}} \\ &\backsim (2 + \sqrt{2})p^2,\end{aligned}$$

and

$$\rho_3 \backsim 1 - \frac{\sqrt{2}}{2}Sp.$$

Thus

$$f_2(K_4) \backsim \frac{2 - \sqrt{2}}{2 + \sqrt{2}}.$$

Hence

$$\max_{K\in[K_{min},K_{max}]} f_2(K) = \max\{f_2(K_{min}), f_2(K_{max}), f_2(K_1), f_2(K_2), f_2(K_3), f_2(K_4)\}$$

$$= \max\{f_2(K_{min}), f_2(K_{max}), 1 - 2\sqrt{2}p^{\frac{1}{2}}q^{\frac{1}{2}}, 1 - 4\sqrt{2}q^{-\frac{1}{2}}S^{\frac{1}{2}}\}.$$

Combining all of the max problems on the four edges, we have that

$$\max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) = \max\{1-2\sqrt{2}p^{\frac{1}{2}}q^{\frac{1}{2}}, 1-4\sqrt{2}q^{-\frac{1}{2}}S^{\frac{1}{2}}, 1-\frac{4X_{min}}{p},$$

$$1 - 4\sqrt{pq}, 1 - 8S^{\frac{1}{2}}q^{-\frac{1}{2}}\}$$

Step 2 of Case 1: Similar as in the previous section, we equibrate the terms and try to solve the following equation

$$2\sqrt{2}\sqrt{pq} = 4\sqrt{2}q^{-1}S^{\frac{1}{2}} = \frac{4X_{min}}{p}.$$

It is equivalent to

$$pq = 4q^{-1}S = \frac{2X_{min}^2}{p^2}.$$

Thus

$$p = 4q^{-2}S.$$

Therefore

$$4q^{-1}S16q^{-4}S^2 = 2X_{min}^2.$$

Hence

$$q \backsim 2S^{\frac{3}{5}}X_{min}^{-\frac{2}{5}}.$$

Therefore

$$p \backsim S^{-\frac{1}{5}}X_{min}^{\frac{4}{5}}.$$

Using the same argument of the previous section, we can prove that the pair $(p_*, q_*) = (S^{-\frac{1}{5}}X_{min}^{\frac{4}{5}}, 2S^{\frac{3}{5}}X_1^{-\frac{2}{5}})$ is the unique solution of our min-max problem

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) = \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p_*,q_*)$$

And

$$\max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p_*,q_*) \backsim 1 - 4X_{min}^{\frac{1}{5}}S^{\frac{1}{5}}.$$

Case 2: $k_{max}L$ is not small and $\omega_{max}L$ is not small and not large Similar as in the previous case, we will consider the problem of finding the maximum on the four edges. We suppose here that $p = C_p S^{-\gamma_p}$, $q = C_q S^{\gamma_q}$ where $0 < \gamma_p, \gamma_q < 1$, $\gamma_p + \gamma_q < 1$ and $2\gamma_q > 1$.

*On the edge $k = k_{max}$, similar as in the previous case, we have that $\rho < C_1 < 1$ where C_1 is a postive constant.

*On the edge $\omega = \omega_{min}$, we have similarly

$$\max_{k\in[k_{min},k_{max}]} \rho(\omega_{min}, k, p, q) = \max\{1 - 4\sqrt{pq}, 1 - 8S^{\frac{1}{2}}q^{-\frac{1}{2}}, 1 - \frac{4X_1}{p}\}.$$

*On the edge $k = k_{min}$, we can have similarly that the equation $h_2(K) = 0$ has four solutions $K_1 \backsim 2\sqrt{2}q^{-\frac{1}{2}}S^{\frac{1}{2}}$, $K_2 \backsim 2\sqrt{2}q^{-1}$, $K_3 \backsim \sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}$, $K_4 \backsim \frac{\sqrt{2}}{2}p$. We can see that in this case $K_2, K_3, K_4 \in [K_{min}, K_{max}]$ and $K_1 \notin [K_{min}, K_{max}]$. Thus

$$\begin{aligned} \max_{K\in[K_{min},K_{max}]} f_2(K) &= \max\{f_2(K_{min}), f_2(K_{max}), f_2(K_2), f_2(K_3), f_2(K_4)\} \\ &= \max\{f_2(K_{min}), f_2(K_{max}), 1 - 2\sqrt{2}p^{\frac{1}{2}}q^{\frac{1}{2}}\}. \end{aligned}$$

*On the edge $\omega = \omega_{max}$, we consider the following cases

+ If $k^2\nu^2$ is much more bigger than $\omega_{max}\nu$ or $k^2\nu^2 \backsim C_k S^{-\gamma_k}$ where $\gamma_k > 1$, we have that

$$\rho(k, \omega_{max}, p, q) \backsim$$

$$\begin{aligned} &\backsim \frac{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 - 2(p + qk^2\nu^2)2k\nu - 2q\omega_{max}\nu\sqrt{2}\omega_{max}\nu}{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 + 2(p + qk^2\nu^2)2k\nu + 2q\omega_{max}\nu\sqrt{2}\omega_{max}\nu} \times \\ &\quad \times \exp(-2k\nu S) \\ &\backsim \frac{4k^2\nu^2 + q^2k^4\nu^4 - 4qk^3\nu^3}{4k^2\nu^2 + q^2k^4\nu^4 + 4qk^3\nu^3}(1 - 2k\nu S) \\ &\backsim 1 - 8q^{-1}k^{-1}\nu^{-1} - 2k\nu S < 1 - 8q^{-\frac{1}{2}}S^{\frac{1}{2}}, \end{aligned}$$

or

$$\rho(k, \omega_{max}, p, q) \backsim$$

$$\begin{aligned} &\backsim \frac{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 - 2(p + qk^2\nu^2)2k\nu - 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}}{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 + 2(p + qk^2\nu^2)2k\nu + 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}} \times \\ &\quad \times \exp(-2k\nu S) \\ &\backsim \frac{4k^2\nu^2 + q^2k^4\nu^4 - 4qk^3\nu^3}{4k^2\nu^2 + q^2k^4\nu^4 + 4qk^3\nu^3}(1 - 2k\nu S) \\ &\backsim 1 - 2qk\nu - 2k\nu S \backsim 1 - 2qk\nu < 1 - 8S^{\frac{1}{2}}q^{-\frac{1}{2}}. \end{aligned}$$

+ If ω_{max} is much more bigger than $k^2\nu^2$ or $k^2\nu^2 \backsim C_k S^{-\gamma_k}$ where $\gamma_k < 1$, we have that

$$\rho(k,\omega_{max},p,q) \backsim$$

$$\begin{aligned}
&\backsim \frac{4\omega_{max}\nu + p^2 + q^2\omega_{max}^2\nu^2 + 2pqk^2\nu^2 - 2(p+qk^2\nu^2)\sqrt{2\omega_{max}\nu} - 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}}{4\omega_{max}\nu + p^2 + q^2\omega_{max}^2\nu^2 + 2pqk^2\nu^2 + 2(p+qk^2\nu^2)\sqrt{2\omega_{max}\nu} + 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}} \times \\
&\quad \times \exp(-\sqrt{2\omega_{max}}S) \\
&\backsim \frac{4\omega_{max}\nu + q^2\omega_{max}^2\nu^2 - 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}}{4\omega_{max}\nu + q^2\omega_{max}^2\nu^2 + 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}}(1-\sqrt{2\omega_{max}}S) \\
&\backsim (1-\sqrt{2}q\sqrt{\omega_{max}\nu})(1-\sqrt{2\omega_{max}}S) \\
&\backsim 1-\sqrt{2}q\sqrt{\omega_{max}\nu} < 1 - 8S^{\frac{1}{2}}q^{-\frac{1}{2}}.
\end{aligned}$$

+ If $k^2\nu^2 \backsim C_k S^{-1}$, we have that

$$\rho(k,\omega_{max},p,q) \backsim$$

$$\begin{aligned}
&\backsim (4\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2qk^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} \\
&\quad -2q\omega_{max}\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2})(4\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + \\
&\quad +2qk^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} + 2q\omega_{max}\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2})^{-1} \times \\
&\quad \times \exp(-\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2 S}) \\
&\backsim 1 - q^{-1}\frac{k^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} + \omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2}}{\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4}} \\
&< 1 - \frac{4X_1}{p}.
\end{aligned}$$

Combining the maximum results on the four edges, we have that

$$\max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} = \max\{1-2\sqrt{2}\sqrt{pq}, 1-\frac{4X_{min}}{p}, 1-4\sqrt{pq}, 1-8S^{\frac{1}{2}}q^{-\frac{1}{2}}\}.$$

Similar as in the previous sections, we solve the equilibrating equation

$$2\sqrt{2}\sqrt{pq} = 8q^{-\frac{1}{2}}S^{\frac{1}{2}} = \frac{4X_{min}}{p}.$$

Thus

$$pq = 8q^{-1}S = \frac{2X_{min}^2}{p^2}.$$

Hence

$$p = 8q^{-2}S.$$

Thus

$$8q^{-1}S64q^{-4}S^2 = 2X_{min}^2.$$

Therefore

$$256q^{-5}S^3 = X_{min}^2.$$

Hence

$$q = 2^{\frac{8}{5}}S^{\frac{3}{5}}X_{min}^{-\frac{2}{5}},$$

and

$$p = 8S2^{-\frac{16}{5}}S^{-\frac{6}{5}}X_{min}^{\frac{4}{5}} = 2^{-\frac{1}{5}}S^{-\frac{1}{5}}X_{min}^{\frac{4}{5}}.$$

Using the same argument of the previous sections, we can prove that the pair $(p_*, q_*) = (S^{-\frac{1}{5}}X_{min}^{\frac{4}{5}}, 2^{\frac{8}{5}}S^{\frac{3}{5}}X_{min}^{-\frac{2}{5}})$ is the unique solution of our min-max problem

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega, k, p, q) = \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega, k, p_*, q_*)$$

And

$$\max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega, k, p_*, q_*) \backsim 1 - 2^{\frac{9}{5}}X_{min}^{\frac{1}{5}}S^{\frac{1}{5}}.$$

3.2.2 Proof of the Theorems in the Nonoverlapping Case

Proof of Theorem 3.2.3 :

Case 1: $\Delta t = C_1\Delta x$, $\Delta y = C_2\Delta x$.

In this case, we have that $\omega_{min} = \frac{\pi}{2T}$, $\omega_{max} = \frac{\pi}{C_1}\Delta x^{-1}$, $k_{min} = \frac{\pi}{2Y}$ and $k_{max} = \frac{\pi}{C_2}\Delta x^{-1}$.

Similar as in the overlapping case, we will consider the max problem on the four edges. We suppose that $p = C_p\Delta x^{-\gamma_p}$, $q = C_q\Delta x^{\gamma_q}$ and $0 < \gamma_p < \gamma_q < 1$.

* On the edge $\omega = \omega_{min}$, similar as in the overlapping case, we put $X = \sqrt{2\sqrt{\omega^2\nu^2 + k^4\nu^4} + 2k^2\nu^2}$, and $\omega\nu = a$, and the convergence factor becomes

$$\rho(\omega, k, p, q) = \frac{\rho_1}{\rho_2} := f_1(X),$$

where

$$\rho_1 = X^2 + \frac{4a^2}{X^2} + p^2 + q^2(X^2 + \frac{4a^2}{X^2})^2\frac{1}{16} - 2(p + q(\frac{X^2}{4} - \frac{a^2}{X^2})X) - q\frac{4a^2}{X} + 2pq(\frac{X^2}{4} - \frac{a^2}{X^2}),$$

and

$$\rho_2 = X^2 + \frac{4a^2}{X^2} + p^2 + q^2(X^2 + \frac{4a^2}{X^2})^2\frac{1}{16} + 2(p + q(\frac{X^2}{4} - \frac{a^2}{X^2})X) + q\frac{4a^2}{X} + 2pq(\frac{X^2}{4} - \frac{a^2}{X^2}).$$

We need to consider the following problem

$$\max_{X\in[X_{min}, X_{max}]} f_1(X),$$

where

$$X_{min} = \sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2},$$

and

$$X_{max} = \sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}.$$

We have that

$$f_1'(X) = g_1(X)h_1(X),$$

where

$$\begin{aligned} g_1(X) &= (-16X^2)(16X^6 + 64a^2X^2 + 16p^2X^4 + q^2X^8 + 8q^2X^2a^2 + 16q^2a^4 + \\ &\quad +32pX^5 + 8qX^7 + 32qa^2X^3 + 8pqX^6 - 32pqX^2a^2)^{-2}, \end{aligned}$$

and

$$\begin{aligned} h_1(X) &= 64p^3X^6 - 64X^8p + 16X^{10}q - q^3X^{12} + 128X^6qa^2 + 768a^2X^4p + \\ &+256a^4X^2q - 4q^2X^{10}p + 4q^3X^8a^2 + 80q^3a^4X^4 + 192q^3a^6 + \\ &+16qX^8p^2 + 192q^2a^4pX^2 - 224q^2X^6pa^2 - 448qa^2X^4p^2. \end{aligned}$$

We can see that

$$\max k \in [k_{min}, k_{max}] f_1(X) = \max\{f_1(X_{min}), f_1(X_{max}), f_1(X_i)\},$$

where $f_1'(X_i) = 0$.

We will try to solve the equation $h_1(X) = 0$.

Similar as in the overlapping case, we can deduce from the equation $h_1(X) = 0$ that

$$0 = 16X^{10}q - q^3X^{12} - 64pX^8 + 64p^3X^6,$$

and we can see that this equation has three solutions $X_1 \backsim 4q^{-1}$, $X_2 \backsim 2p^2q^{-\frac{1}{2}}$, and $X_3 \backsim p$ and $X_1, X_2, X_3 \in [X_{min}, X_{max}]$.

For $f_1(X_1)$, we have that

$$\begin{aligned} \rho_1 &\backsim 16q^{-2} + \frac{a^2q^2}{4} + p^2 + q^2(16q^{-2} + \frac{a^2q^2}{4})^2\frac{1}{16} - 2(p + q(4q^{-2} - \frac{a^2q^2}{16}))4q^{-1} \\ &\quad -a^2q^2 + 2pq(4q^{-2} - \frac{a^2q^2}{16}) + 8pq^{-1} \\ &\backsim 16q^{-2} + p^2 + 16q^{-2} - 8pq^{-1} - 32q^{-2} + 8pq^{-1} \backsim p^2, \end{aligned}$$

and

$$\begin{aligned} \rho_1 &\backsim 16q^{-2} + \frac{a^2q^2}{4} + p^2 + q^2(16q^{-2} + \frac{a^2q^2}{4})^2\frac{1}{16} + 2(p + q(4q^{-2} - \frac{a^2q^2}{16}))4q^{-1} \\ &\quad +a^2q^2 + 2pq(4q^{-2} - \frac{a^2q^2}{16}) + 8pq^{-1} \\ &\backsim 16q^{-2} + p^2 + 16q^{-2} + 8pq^{-1} + 32q^{-2} + 8pq^{-1} \backsim 64q^{-2}. \end{aligned}$$

Thus

$$f_1(X_1) \backsim \frac{p^2q^2}{64}.$$

For $f_1(X_2)$, we have that

$$\begin{aligned}
\rho_1 &\backsim 4pq^{-1} + a^2p^{-1}q + p^2 + q^2(4pq^{-1} + a^2p^{-1}q)^2\frac{1}{16} \\
&\quad -2(p + q(pq^{-1} - \frac{a^2p^{-1}q}{4}))2p^{\frac{1}{2}}q^{-\frac{1}{2}} - 2a^2p^{-\frac{1}{2}}q^{\frac{3}{2}} + 2pq(pq^{-1} - \frac{a^2p^{-1}q}{4}) \\
&\backsim 4pq^{-1} + p^2 + p^2 - 8p^{\frac{3}{2}}q^{-\frac{1}{2}} + 2p^2 \\
&\backsim 4pq^{-1} + 4p^2 - 8p^{\frac{3}{2}}q^{-\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\rho_2 &\backsim 4pq^{-1} + a^2p^{-1}q + p^2 + q^2(4pq^{-1} + a^2p^{-1}q)^2\frac{1}{16} \\
&\quad +2(p + q(pq^{-1} - \frac{a^2p^{-1}q}{4}))2p^{\frac{1}{2}}q^{-\frac{1}{2}} + 2a^2p^{-\frac{1}{2}}q^{\frac{3}{2}} + 2pq(pq^{-1} - \frac{a^2p^{-1}q}{4}) \\
&\backsim 4pq^{-1} + p^2 + p^2 + 8p^{\frac{3}{2}}q^{-\frac{1}{2}} + 2p^2 \\
&\backsim 4pq^{-1} + 4p^2 + 8p^{\frac{3}{2}}q^{-\frac{1}{2}}.
\end{aligned}$$

Thus

$$\rho \backsim \frac{4pq^{-1} + 4p^2 - 8p^{\frac{3}{2}}q^{-\frac{1}{2}}}{4pq^{-1} + 4p^2 + 8p^{\frac{3}{2}}q^{-\frac{1}{2}}} \backsim 1 - 4p^{\frac{1}{2}}q^{\frac{1}{2}}.$$

For $f_1(X_3)$, we have that

$$\begin{aligned}
\rho_1 &\backsim p^2 + 4a^2p^{-2} + p^2 + q^2(p^2 + 4a^2p^{-2})^2\frac{1}{16} \\
&\quad -2(p + q(\frac{p^2}{4} - a^2p^{-2}))p - 4a^2qp^{-2} + 2pq(\frac{p^2}{4} - a^2p^{-2}) \\
&\backsim 2p^2 + \frac{q^2p^4}{16} - 2p^2 - \frac{qp^3}{2} + \frac{qp^3}{2} \backsim \frac{q^2p^4}{16},
\end{aligned}$$

$$\begin{aligned}
\rho_2 &\backsim p^2 + 4a^2p^{-2} + p^2 + q^2(p^2 + 4a^2p^{-2})^2\frac{1}{16} \\
&\quad +2(p + q(\frac{p^2}{4} - a^2p^{-2}))p + 4a^2qp^{-2} + 2pq(\frac{p^2}{4} - a^2p^{-2}) \\
&\backsim 2p^2 + \frac{q^2p^4}{16} + 2p^2 + \frac{qp^3}{2} + \frac{qp^3}{2} \backsim \frac{q^2p^4}{16} + 4p^2 + qp^3.
\end{aligned}$$

Thus

$$f_1(X_3) \backsim \frac{\frac{q^2p^4}{16}}{\frac{q^2p^4}{16}+4p^2+qp^3} \backsim \frac{p^2q^2}{64}.$$

For $f_1(X_{min})$, we have that

$$\rho_1 \backsim p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) - 2pX_{min},$$

and

$$\rho_2 \backsim p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) + 2pX_{min}.$$

Thus

$$\begin{aligned} f_1(X_{min}) &= \frac{\rho_1}{\rho_2} \backsim \frac{p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) - 2pX_{min}}{p^2 + 2pq(\frac{X_{min}^2}{4} - \frac{a^2}{X_{min}^2}) + 2pX_{min}}(1 - X_{min}S) \\ &\backsim 1 - \frac{4X_{min}}{p}. \end{aligned}$$

Thus

$$\begin{aligned} \max_{X\in[X_{min},X_{max}]} f_1(X) &= \max\{f_1(X_{min}), f_1(X_{max}), f_1(X_1), f_1(X_2), f_1(X_3)\} \\ &= \max\{1 - \frac{4X_{min}}{p}, 1 - 4p^{\frac{1}{2}}q^{-\frac{1}{2}}, f_1(X_{max})\}. \end{aligned}$$

* On the edge $k = k_{min}$, similar as in the overlapping case, we put $K = \sqrt{2\sqrt{\omega^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}$, and $a = k_{min}^2\nu^2$, then

$$K \in [K_{min}, K_{max}] =$$

$$= [\sqrt{2\sqrt{\omega_{min}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}, \sqrt{2\sqrt{\omega_{max}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}].$$

Thus

$$f_2(K) = \rho(\omega, k, p, q) = \frac{\rho_1}{\rho_2},$$

where

$$\rho_1 = 4(\frac{K^2}{2} - a) + p^2 + q^2(\frac{K^2}{2} - a)^2 + 2pqa - 2(p + qa)K - 2q(\frac{K^3}{2} - 2Ka),$$

and

$$\rho_2 = 4(\frac{K^2}{2} - a) + p^2 + q^2(\frac{K^2}{2} - a)^2 + 2pqa + 2(p+qa)K + 2q(\frac{K^3}{2} - 2Ka).$$

The max problem turns into the following problem

$$\max_{K \in [K_{min}, K_{max}]} f_2(K).$$

We can see that

$$\max_{K \in [K_{min}, K_{max}]} f_2(K) = \max\{f_2(K_{min}), f_2(K_{max}), f_2(K_i)\},$$

where K_i's are the points where $f_2'(K_i) = 0$.

We have that

$$f_2'(K) = g_2(K)h_2(K),$$

where

$$g_2(K) = 8(8K^2 - 16a + 4p^2 + q^2K^4 - 4q^2K^2a + 4q^2a^2 + 8Kp - 8Kqa + 4qK^3 + 8pqa)^{-2},$$

and

$$\begin{aligned} h_2(K) \quad = \quad & -12qK^2p^2 + 6q^2K^4p + 8pq^2a + 32K^2qa - 2q^3K^4a - 8p^2qa - 32q^2K^2ap \\ & +16K^2p - 8qK^4 - 4q^3K^2a^2 - 8p^3 + q^3K^6 + 32pa - 32qa^2 + 8q^3a^3. \end{aligned}$$

From the equation $h_2(K) = 0$, we get

$$K^6q^3 - 8qK^4 + K^2 16p - 8p^3 = 0.$$

The equation has three solutions $K_1 \backsim 2\sqrt{2}q^{-1}$, $K_2 \backsim \sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}$, $K_3 \backsim \frac{\sqrt{2}}{2}p$.

For $f_2(K_1)$, we have

$$\begin{aligned} \rho_1 \quad \backsim \quad & 16q^{-2} - 4a + p^2 + q^2(4q^{-2} - a)^2 - 2(p+qa)2\sqrt{2}q^{-1} \\ & -2q(8\sqrt{2}q^{-3} - 4\sqrt{2}q^{-1}a) + 2pqa \\ \backsim \quad & 16q^{-2} + p^2 + 16q^{-2} - 4\sqrt{2}pq^{-1} - 16\sqrt{2}q^{-2} \backsim 16(2 - \sqrt{2})q^{-2}, \end{aligned}$$

and

$$\begin{aligned} \rho_2 \quad \backsim \quad & 16q^{-2} - 4a + p^2 + q^2(4q^{-2} - a)^2 + 2(p+qa)2\sqrt{2}q^{-1} \\ & +2q(8\sqrt{2}q^{-3} - 4\sqrt{2}q^{-1}a) + 2pqa \\ \backsim \quad & 16q^{-2} + p^2 + 16q^{-2} + 4\sqrt{2}pq^{-1} + 16\sqrt{2}q^{-2} \backsim 16(2 + \sqrt{2})q^{-2}. \end{aligned}$$

Thus

$$f_2(K_1) \backsim \frac{2-\sqrt{2}}{2+\sqrt{2}}.$$

For $f_2(K_2)$, we have

$$\begin{aligned}
\rho_1 &\backsim 4pq^{-1} - 4a + p^2 + q^2(pq^{-1} - a)^2 + 2pqa - 2(p+qa)\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}} \\
&\quad -2q(\sqrt{2}p^{\frac{3}{2}}q^{-\frac{3}{2}} - 2a\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}) \\
&\backsim 4pq^{-1} + p^2 + p^2 - 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} - 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} \backsim 4pq^{-1} + 2p^2 - 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\rho_2 &\backsim 4pq^{-1} - 4a + p^2 + q^2(pq^{-1} - a)^2 + 2pqa + 2(p+qa)\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}} \\
&\quad +2q(\sqrt{2}p^{\frac{3}{2}}q^{-\frac{3}{2}} - 2a\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}}) \\
&\backsim 4pq^{-1} + p^2 + p^2 + 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} + 2\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}} \backsim 4pq^{-1} + 2p^2 + 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}}.
\end{aligned}$$

Thus

$$f_2(K_2) \backsim \frac{4pq^{-1} + 2p^2 - 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}}}{4pq^{-1} + 2p^2 + 4\sqrt{2}p^{\frac{3}{2}}q^{-\frac{1}{2}}} \backsim 1 - 2\sqrt{2}\sqrt{pq}.$$

For $f_2(K_3)$, we have that

$$\begin{aligned}
\rho_1 &\backsim p^2 - 4a + p^2 + q^2(\frac{p^2}{4} - a)^2 - 2(p+qa)\frac{p}{\sqrt{2}} - 2q(\frac{p^3}{4\sqrt{2}} - \sqrt{2}pa) + 2pqa \\
&\backsim 2p^2 + \frac{q^2p^4}{16} - \sqrt{2}p^2 - \frac{qp^3}{2\sqrt{2}} \backsim (2-\sqrt{2})p^2,
\end{aligned}$$

and

$$\begin{aligned}
\rho_2 &\backsim p^2 - 4a + p^2 + q^2(\frac{p^2}{4} - a)^2 + 2(p+qa)\frac{p}{\sqrt{2}} + 2q(\frac{p^3}{4\sqrt{2}} - \sqrt{2}pa) + 2pqa \\
&\backsim 2p^2 + \frac{q^2p^4}{16} + \sqrt{2}p^2 + \frac{qp^3}{2\sqrt{2}} \backsim (2+\sqrt{2})p^2.
\end{aligned}$$

Thus

$$f_2(K_3) \backsim \frac{2-\sqrt{2}}{2+\sqrt{2}}.$$

We suppose that $K_2 = \sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}} \in [K_{min}, K_{max}]$. Thus

$$\max_{K\in[K_{min},K_{max}]} = \max\{f_2(K_{max}), 1 - \frac{4X_{min}}{p}, 1 - 2\sqrt{2}\sqrt{pq}\}.$$

* On the edge $k = k_{max}$, we have that

$$\sqrt{\omega^2\nu^2 + k_{max}^4\nu^4} \backsim k_{max}^2\nu^2.$$

Thus $\rho(\omega, k_{max}, p, q) \backsim$

$$\backsim \frac{4k_{max}^2\nu^2 + p^2 + q^2k_{max}^4\nu^4 + 2pqk_{max}^2\nu^2 - 4(p + qk_{max}^2\nu^2)k_{max}\nu - 2\sqrt{2}q(\omega\nu)^{\frac{3}{2}}}{4k_{max}^2\nu^2 + p^2 + q^2k_{max}^4\nu^4 + 2pqk_{max}^2\nu^2 + 4(p + qk_{max}^2\nu^2)k_{max}\nu + 2\sqrt{2}q(\omega\nu)^{\frac{3}{2}}}$$
$$\backsim \frac{q^2k_{max}^4\nu^4 - 4qk_{max}^3\nu^3}{q^2k_{max}^4\nu^4 + 4qk_{max}^3\nu^3} \backsim 1 - 8q^{-1}k_{max}^{-1}\nu^{-1}.$$

*On the edge $\omega = \omega_{max}$, we consider the following cases

If $k^2\nu^2$ is much more bigger than $\omega_{max}\nu$ or $k^2\nu^2 \backsim C_kS^{-\gamma_k}$ where $\gamma_k > 1$, we have that $\rho(k, \omega_{max}, p, q) \backsim$

$$\backsim \frac{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 - 2(p + qk^2\nu^2)2k\nu - 2q\omega_{max}\nu\sqrt{2}\omega_{max}\nu}{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 + 2(p + qk^2\nu^2)2k\nu + 2q\omega_{max}\nu\sqrt{2}\omega_{max}\nu}$$
$$\backsim \frac{4k^2\nu^2 + q^2k^4\nu^4 - 4qk^3\nu^3}{4k^2\nu^2 + q^2k^4\nu^4 + 4qk^3\nu^3}$$
$$\backsim 1 - 8q^{-1}k^{-1}\nu^{-1} \leq 1 - 8q^{-1}k_{max}^{-1}\nu^{-1} = \rho(\omega_{max}, k_{max}, p, q),$$

or $\rho(k, \omega_{max}, p, q) \backsim$

$$\backsim \frac{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 - 2(p + qk^2\nu^2)2k\nu - 2q\omega_{max}\nu\sqrt{2}\omega_{max}\nu}{4k^2\nu^2 + p^2 + q^2k^4\nu^4 + 2pqk^2\nu^2 + 2(p + qk^2\nu^2)2k\nu + 2q\omega_{max}\nu\sqrt{2}\omega_{max}\nu}\exp(-2k\nu S)$$
$$\backsim \frac{4k^2\nu^2 + q^2k^4\nu^4 - 4qk^3\nu^3}{4k^2\nu^2 + q^2k^4\nu^4 + 4qk^3\nu^3}$$
$$\backsim 1 - 2qk\nu < 1 - 4\sqrt{pq}.$$

If ω_{max} is much more bigger than $k^2\nu^2$ or $k^2\nu^2 \backsim C_kS^{-\gamma_k}$ where $\gamma_k < 1$, we have that $\rho(k, \omega_{max}, p, q) \backsim$

$$\backsim \frac{4\omega_{max}\nu + p^2 + q^2\omega_{max}^2\nu^2 + 2pqk^2\nu^2 - 2(p + qk^2\nu^2)\sqrt{2\omega_{max}\nu} - 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}}{4\omega_{max}\nu + p^2 + q^2\omega_{max}^2\nu^2 + 2pqk^2\nu^2 + 2(p + qk^2\nu^2)\sqrt{2\omega_{max}\nu} + 2q\omega_{max}\nu\sqrt{2\omega_{max}\nu}}$$
$$\backsim \frac{4\omega_{max}\nu + q^2\omega_{max}^2\nu^2 - 2(p + q\omega_{max}\nu)\sqrt{2\omega_{max}\nu}}{4\omega_{max}\nu + q^2\omega_{max}^2\nu^2 + 2(p + q\omega_{max}\nu)\sqrt{2\omega_{max}\nu}}$$
$$\backsim 1 - \sqrt{2}(p + q\omega_{max}\nu)(\omega_{max}\nu)^{-\frac{1}{2}} \leq 1 - 2\sqrt{2pq},$$

here, we use the assumptions: $0 < \gamma_p, \gamma_q < 1$, $\gamma_p + \gamma_q < 1$ and $2\gamma_q > 1$.

If $k^2\nu^2 \backsim C_k S^{-1}$, we have that $\rho(k, \omega_{max}, p, q) \backsim$

$$\begin{aligned}
&\backsim (4\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2(p + qk^2)\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} \\
&\quad -2q\omega_{max}\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2})(4\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} \\
&\quad -2(p + qk^2\nu^2)\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} - 2q\omega_{max}\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2})^{-1} \\
&\backsim 1 - \frac{(p + qk^2\nu^2)\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} + q\omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2}}{\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4}} \\
&\leq 1 - 2\sqrt{2pq}.
\end{aligned}$$

Combining the maximum results on the four edges, we have that

$$\max_{\omega \in [\omega_{min}, \omega_{max}], k \in [k_{min}, k_{max}]} \rho(\omega, k, p, q) = \max\{1 - \frac{4X_{min}}{p}, 1 - 2\sqrt{2pq}, 1 - 4\sqrt{pq}, 1 - 8q^{-1}k_{max}^{-1}\nu^{-1}\}.$$

Similar as in previous sections, we solve the equilibrating equation

$$2\sqrt{2pq} = \frac{4X_{min}}{p} = 8q^{-1}k_{max}^{-1}\nu^{-1}.$$

Thus

$$p^3 q = 2X_{min}^2,$$

and

$$pq^3 = 8k_{max}^{-2}\nu^{-2}.$$

Hence

$$pq = 2X_{min}^{\frac{1}{2}}k_{max}^{-\frac{1}{2}}\nu^{-\frac{1}{2}}.$$

Thus

$$p^2 = X_{min}^{\frac{3}{2}}k_{max}^{\frac{1}{2}}\nu^{\frac{1}{2}}.$$

And then

$$p = X_{min}^{\frac{3}{4}}k_{max}^{\frac{1}{4}}\nu^{\frac{1}{4}} = X_{min}^{\frac{3}{4}}(\frac{\pi\nu}{C_2})^{\frac{1}{4}}\Delta x^{-\frac{1}{4}}.$$

Therefore

$$q = 2X_{min}^{-\frac{1}{4}}k_{max}^{-\frac{3}{4}}\nu^{-\frac{3}{4}} = 2X_{min}^{-\frac{1}{4}}(\frac{\pi\nu}{C_2})^{-\frac{3}{4}}\Delta x^{\frac{3}{4}}.$$

Checking with the condition $\sqrt{2}p^{\frac{1}{2}}q^{-\frac{1}{2}} < \sqrt{2\sqrt{\omega_{max}^2\nu^2 + k_{min}^4\nu^4} + 2k_{min}^2\nu^2}$, we can see that since $\frac{X_{min}C_1}{C_2} < 2$, we have $\frac{X_{min}k_{max}\nu}{2} < \omega_{max}\nu$, or the condition is satisfied in this case.

Using the same argument of the previous section, we can prove that the pair $(p_*, q_*) = (X_{min}^{\frac{3}{4}} k_{max}^{\frac{1}{4}} \nu^{\frac{1}{4}}, 2X_{min}^{-\frac{1}{4}} k_{max}^{-\frac{3}{4}} \nu^{-\frac{3}{4}})$ is the unique solution of our min-max problem

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) = \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p_*,q_*)$$

And

$$\max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p_*,q_*) \backsim 1-4X_{min}^{\frac{1}{4}} k_{max}^{-\frac{1}{4}} \nu^{-\frac{1}{4}} = 4X_{min}^{\frac{1}{4}} (\frac{\pi\nu}{C_2})^{-\frac{1}{4}} \Delta x^{\frac{1}{4}}.$$

Case 2: $\Delta t = C_1 \Delta x^2$, $\Delta y = C_2 \Delta x$.

Similar as in Case 1, we consider the max problem on the four edges.

* On the edge $\omega = \omega_{min}$, similar as in case 1, we also have that

$$\begin{aligned} \max_{X\in[X_{min},X_{max}]} f_1(X) &= \max\{f_1(X_{min}), f_1(X_{max}), f_1(X_1), f_1(X_2), f_1(X_3)\} \\ &= \max\{1 - \frac{4X_{min}}{p}, 1 - 4p^{\frac{1}{2}} q^{-\frac{1}{2}}, f_1(X_{max})\}. \end{aligned}$$

* On the edge $k = k_{min}$, similar as in Case 1, we also have

$$\max_{K\in[K_{min},K_{max}]} f_2(K) = \max\{f_2(K_{max}), 1 - \frac{4X_{min}}{p}, 1 - 2\sqrt{2}\sqrt{pq}\},$$

with the remark that in this case

$$K_2 = \sqrt{2} p^{\frac{1}{2}} q^{-\frac{1}{2}} < \sqrt{2\sqrt{\omega_{max}^2 \nu^2 + k_{min}^4 \nu^4} + 2k_{min}^2 \nu^2}.$$

* On the edge $k = k_{max}$:

If $\omega\nu$ is much smaller than $k_{max}^2 \nu^2$ or $\omega\nu \backsim C_\omega \Delta x^{-\gamma}$ where $0 \leq \gamma < 2$, then

$$\begin{aligned} \rho &\backsim \frac{4k_{max}^2\nu^2 + p^2 + q^2 k_{max}^4 \nu^4 + 2pq k_{max}^2 \nu^2 - 2q k_{max}^2 \nu^2 2k_{max}\nu - 2q\omega\nu\sqrt{\omega\nu}}{4k_{max}^2\nu^2 + p^2 + q^2 k_{max}^4 \nu^4 + 2pq k_{max}^2 \nu^2 + 2q k_{max}^2 \nu^2 2k_{max}\nu + 2q\omega\nu\sqrt{\omega\nu}} \\ &\backsim \frac{q^2\nu^2 k_{max}^4 \nu^4 - 4q k_{max}^2 \nu^2 k_{max}\nu}{q^2\nu^2 k_{max}^4 \nu^4 + 4q k_{max}^2 \nu^2 k_{max}\nu} \backsim 1 - 8q^{-1} k_{max}^{-1} \nu^{-1}. \end{aligned}$$

If $\omega\nu \backsim C_\omega \Delta x^{-2}$, then

$$\begin{aligned}
\rho &\backsim (q^2(k_{max}^4\nu^4+\omega^2\nu^2) - 2qk_{max}^2\nu^2\sqrt{2\sqrt{\omega^2\nu^2+k_{max}^4\nu^4}+2k_{max}^2\nu^2} \\
&\quad -2q\omega\nu\sqrt{2\sqrt{\omega^2\nu^2+k_{max}^4\nu^4}-2k_{max}^2\nu^2})(q^2(k_{max}^4\nu^4+\omega^2\nu^2) \\
&\quad -2qk_{max}^2\nu^2\sqrt{2\sqrt{\omega^2\nu^2+k_{max}^4\nu^4}+2k_{max}^2\nu^2}+2q\omega\nu\sqrt{2\sqrt{\omega^2\nu^2+k_{max}^4\nu^4}+2k_{max}^2\nu^2})^{-1} \\
&\backsim 1-4q^{-1}\frac{k_{max}^2\nu^2\sqrt{2\sqrt{\omega^2\nu^2+k_{max}^4\nu^4}+2k_{max}^2\nu^2}+\omega\nu\sqrt{2\sqrt{\omega^2\nu^2+k_{max}^4\nu^4}-2k_{max}^2\nu^2}}{\omega^2\nu^2+k_{max}^4\nu^4} \\
&= 1-4q^{-1}k_{max}^{-1}\nu^{-1}\frac{\sqrt{2\sqrt{a^2+1}+2a}+a\sqrt{2\sqrt{a^2+1}-2a}}{a^2+1},
\end{aligned}$$

where $a = \frac{\omega\nu}{k_{max}^2\nu^2}$.

Since $H(a) = \frac{\sqrt{2\sqrt{a^2+1}+2a}+a\sqrt{2\sqrt{a^2+1}-2a}}{a^2+1}$ is a decreasing function. We can see that in this case $\rho(\omega, k_{max}, p, q) \leq \rho(\omega_{max}, k_{max}, p, q)$. And moreover, $\rho(\omega_{max}, k_{max}, p, q) \backsim$

$$1-4q^{-1}\frac{k_{max}^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2+k_{max}^4\nu^4}+2k_{max}^2\nu^2}+\omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2+k_{max}^4\nu^4}-2k_{max}^2\nu^2}}{\omega_{max}^2\nu^2+k_{max}^4\nu^4}.$$

* On the edge $\omega = \omega_{max}$:

If $\omega_{max}\nu$ is much more larger that $k^2\nu^2$ or $k^2\nu^2 \backsim C_k\Delta^{-\gamma}$ where $0 \leq \gamma < 2$, we have that

$$\begin{aligned}
\rho &\backsim \frac{4\omega_{max}\nu + q^2\omega_{max}^2\nu^2 - 2(p+qk^2\nu^2)\sqrt{2\omega_{max}\nu} - 2\sqrt{2}q(\omega_{max}\nu)^{\frac{3}{2}}}{4\omega_{max}\nu + q^2\omega_{max}^2\nu^2 + 2(p+qk^2\nu^2)\sqrt{2\omega_{max}\nu} + 2\sqrt{2}q(\omega_{max}\nu)^{\frac{3}{2}}} \\
&\backsim \frac{q^2\omega_{max}^2\nu^2 - 2\sqrt{2}q(\omega_{max}\nu)^{\frac{3}{2}}}{q^2\omega_{max}^2\nu^2 + 2\sqrt{2}q(\omega_{max}\nu)^{\frac{3}{2}}} \backsim 1-4\sqrt{2}q^{-1}\omega_{max}^{-\frac{1}{2}}\nu^{-\frac{1}{2}}.
\end{aligned}$$

If $k^2\nu^2 \backsim C_k \Delta x^{-2}$, we have that $\rho \backsim$

$$\begin{aligned}
& (q^2(k^4\nu^4 + \omega_{max}^2\nu^2) - 2qk^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} \\
& -2q\omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2})(q^2(k^4\nu^4 + \omega_{max}^2\nu^2) \\
& +2qk^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} + 2q\omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2})^{-1} \\
\backsim \quad & 1 - 4q^{-1}\frac{k^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} + 2k^2\nu^2} + \omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k^4\nu^4} - 2k^2\nu^2}}{\omega_{max}^2\nu^2 + k^4\nu^4} \\
\leq \quad & 1 - 4q^{-1}A,
\end{aligned}$$

where

$$A =$$

$$\min\{\sqrt{2}, \frac{k_{max}^2\nu^2\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k_{max}^4\nu^4} + 2k_{max}^2\nu^2} + \omega_{max}\nu\sqrt{2\sqrt{\omega_{max}^2\nu^2 + k_{max}^4\nu^4} - 2k_{max}^2\nu^2}}{(\omega_{max}^2\nu^2 + k_{max}^4\nu^4)\sqrt{\omega_{max}\nu}}\}$$

Combining the maximum results on the four edges, we have that

$$\max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega, k, p, q) =$$

$$= \max\{1 - \frac{4X_{min}}{p}, 1 - 4\sqrt{pq}, 1 - 2\sqrt{2pq}, 1 - 4q^{-1}\omega_{max}^{-\frac{1}{2}}\nu^{-\frac{1}{2}}A\}.$$

Similar as in the previous sections, we solve the equilibrating equation

$$2\sqrt{2pq} = 4q^{-1}(\omega_{max}\nu)^{-\frac{1}{2}}A = \frac{4X_{min}}{p}.$$

We have that

$$p^3q = 2X_{min}^2,$$

and

$$pq^3 = 2(\omega_{max}\nu)^{-1}A^2.$$

Thus

$$pq = \sqrt{2}X_{min}^{\frac{1}{2}}(\omega_{max}\nu)^{-\frac{1}{4}}A^{\frac{1}{2}}.$$

Hence

$$p^2 = \sqrt{2}X_{min}^{\frac{3}{2}}(\omega_{max}\nu)^{\frac{1}{4}}A^{-\frac{1}{2}}.$$

Thus

$$p = 2^{\frac{1}{4}} X_{min}^{\frac{3}{4}} (\omega_{max}\nu)^{\frac{1}{8}} A^{-\frac{1}{4}} = 2^{\frac{1}{4}} X_{min}^{\frac{3}{4}} (\frac{\pi\nu}{C_1})^{\frac{1}{8}} A^{-\frac{1}{4}} \Delta x^{-\frac{1}{4}},$$

and

$$q = 2^{\frac{1}{4}} X_{min}^{-\frac{1}{4}} (\omega_{max})^{-\frac{3}{8}} A^{\frac{3}{4}} = 2^{\frac{1}{4}} X_{min}^{-\frac{1}{4}} (\frac{\pi\nu}{C_1})^{-\frac{3}{8}} A^{\frac{3}{4}} \Delta x^{\frac{3}{4}}.$$

Using the same argument of the previous section, we can prove that the pair $(p_*, q_*) = (X_{min}^{\frac{3}{4}} k_{max}^{\frac{1}{4}} \nu^{\frac{1}{4}}, 2X_{min}^{-\frac{1}{4}} k_{max}^{-\frac{3}{4}} \nu^{-\frac{3}{4}})$ is the unique solution of our min-max problem

$$\min_{p,q\geq 0} \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p,q) = \max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p_*,q_*)$$

And

$$\max_{\omega\in[\omega_{min},\omega_{max}],k\in[k_{min},k_{max}]} \rho(\omega,k,p_*,q_*) \backsim 1 - 2^{\frac{7}{4}} X_{min}^{-\frac{3}{4}} \omega_{max}^{-\frac{1}{8}} \nu^{-\frac{1}{8}} A^{\frac{1}{4}} = 1 - 2^{\frac{7}{4}} X_{min}^{-\frac{3}{4}} (\frac{\pi\nu}{C_2})^{-\frac{1}{8}} A^{\frac{1}{4}} \Delta x^{\frac{1}{4}}.$$

3.3 Numerical Results

In this section, we intend to verify our theoretical results on the optimized parameters for the optimized Schwarz methods obtained in the previous sections. We chose for the problem parameters $\nu = 1$, in the domain $[-1, 1] \times [0, 1]$, $T = 1$ with homogeneous boundary conditions. We discretize the problem with Euler backward scheme and use random initial conditions.

3.3.1 Test 1

First of all, we would like to compare the behavior of classical Schwarz method and optimized Schwarz methods, with Robin and Ventcell transmission conditions in both cases: nonoverlapping and overlapping. We choose 30 grid points on both the time interval and the space interval. We choose the overlapping length to be 1 grid points for overlapping algorithms and we compute the solution in 10 iterations. We choose the parameter p for the Robin transmission condition to be our computed optimal p and the parameter (p, q) for the first order transmission condition to be our computed optimal (p, q). We can see that the optimized Schwarz methods converge much faster than the classical one and the otimized Schwarz with the optimal first order transmission condition converges faster than the optimal Robin one as in Figure 3.3.1.

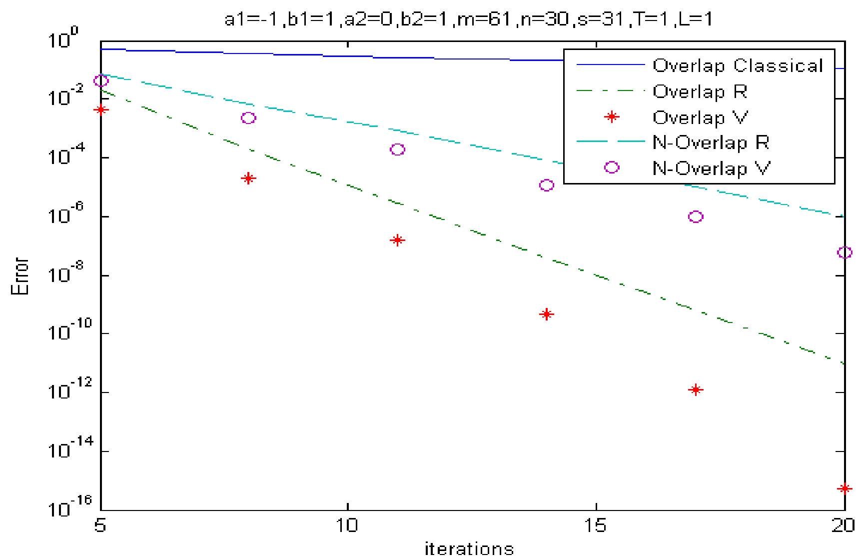

a1=-1,b1=1,a2=0,b2=1,m=61,n=30,s=31,T=1,L=1
Error
iterations
Overlap Classical
Overlap R
Overlap V
N-Overlap R
N-Overlap V

Figure 3.3.1

3.3.2 Test 2

Secondly, we would like to test the accuracy of our theoretical optimized Robin parameters. According to our theoretical results, the optimized parameters depend on the constans C, D in both cases $dt = Cdx$, $dt = Ddy$ and $dt = Cdx^2$, $dt = Ddy^2$. Thus in order to vary C and D, we vary the number of grid points on T, x and y directions to see how the algorithms behave.

In this test we solve solve by domain decomposition methods with Robin transmission conditions, and Euleur backward scheme, the the heat equation in 2 D, $\nu = 1$, in a domain $[-1, 1] \times [0, 1]$, $T = 1$, 10 iterations, the overlapping size is one grid point. We will keep the same space-time window and observe the behavior of the optimal p when we vary the number of grid points on both space and time.

In the first case (Figure 3.3.2.A), we choose 50 grid points for the x direction, 50 grid points for the y direction, and 200 grid points for the T direction.

In the second case (Figure 3.3.2.B), we choose 100 grid points for the x direction, 100 grid points for the y direction, and 20 grid points for the T direction.

In the third case (Figure 3.3.2.C), we choose 50 grid points for the x direction, 50 grid points for the y direction, and 20 grid points for the T direction.

In the forth case (Figure 3.3.2.D), we choose 200 grid points for the x direction, 100 grid points for the y direction, and 100 grid points for the T direction.

In the fifth case (Figure 3.3.2.E), we choose 200 grid points for the x direction, 90 grid points for the y direction, and 20 grid points for the T direction.

In the sixth case (Figure 3.3.2.F), we choose 60 grid points for the x direction, 50 grid points for the y direction, and 300 grid points for the T direction.

In the seventh case (Figure 3.3.2.G), we choose 40 grid points for the x direction, 80 grid points for the y direction, and 300 grid points for the T direction.

In the eighth case (Figure 3.3.2.H), we choose 80 grid points for the x direction, 60 grid points for the y direction, and 20 grid points for the T direction.

In the nineth case (Figure 3.3.2.I), we choose 120 grid points for the x direction, 50 grid points for the y direction, and 60 grid points for the T direction.

In the tenth case (Figure 3.3.2.J), we choose 30 grid points for the x direction, 15 grid points for the y direction, and 60 grid points for the T direction.

We can see that in most of the case, the theoretical optimal p (the star $*$ on t

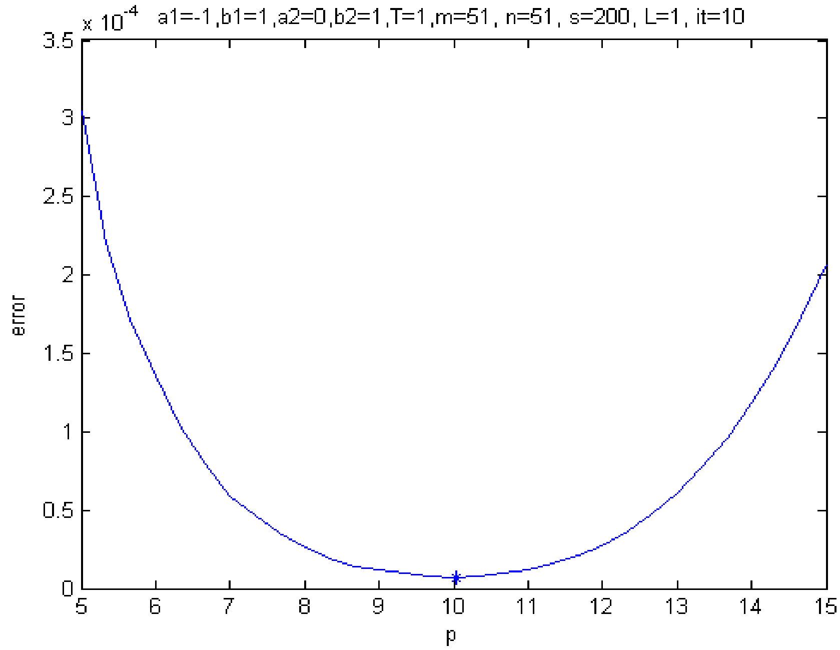


Figure 3.3.2.A.

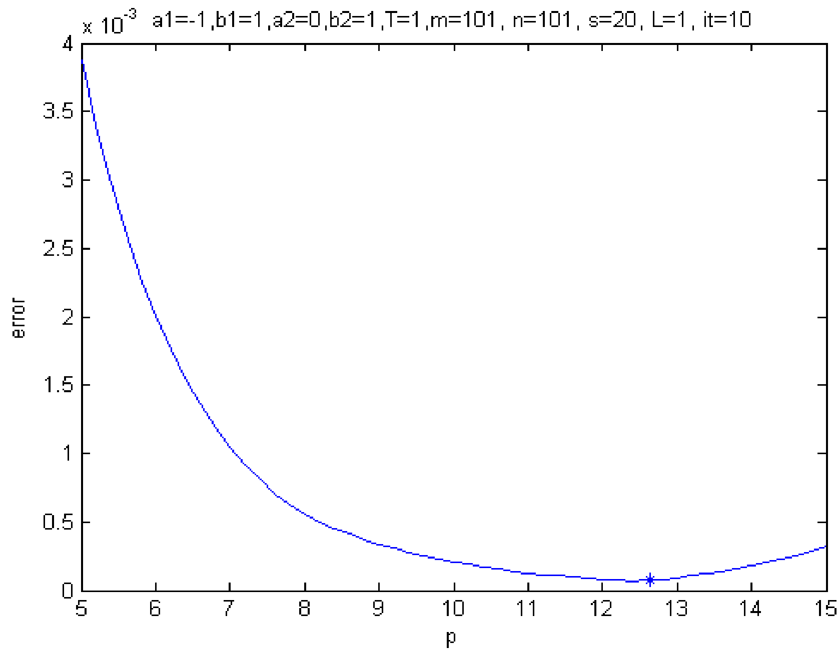

x 10^-3 a1=-1,b1=1,a2=0,b2=1,T=1,m=101, n=101, s=20, L=1, it=10
4
3.5
3
2.5
2
1.5
1
0.5
0
error
5
6
7
8
9
10
11
12
13
14
15
p

Figure 3.3.2.B.

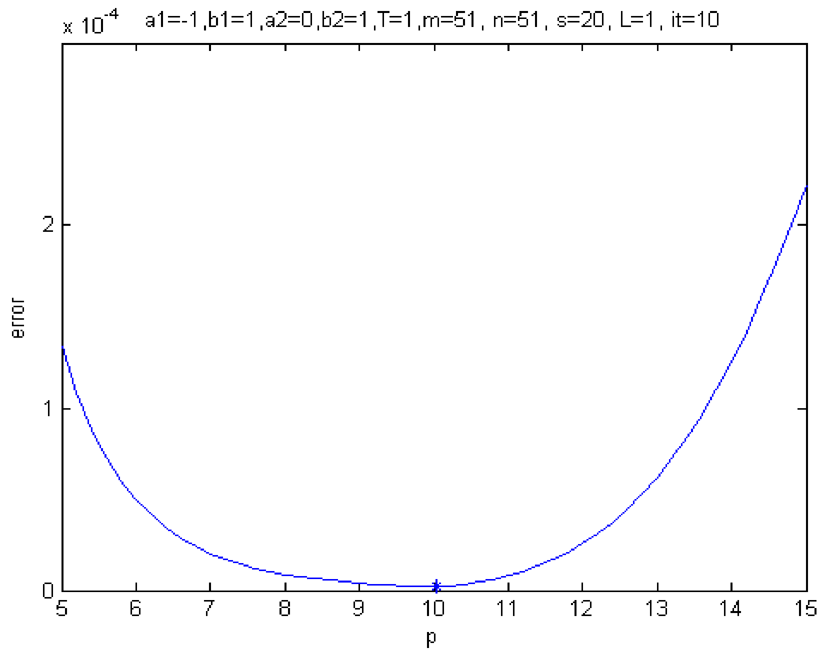

x 10^-4 a1=-1,b1=1,a2=0,b2=1,T=1,m=51, n=51, s=20, L=1, it=10
error
p
0
1
2
5
6
7
8
9
10
11
12
13
14
15

Figure 3.3.2.C.

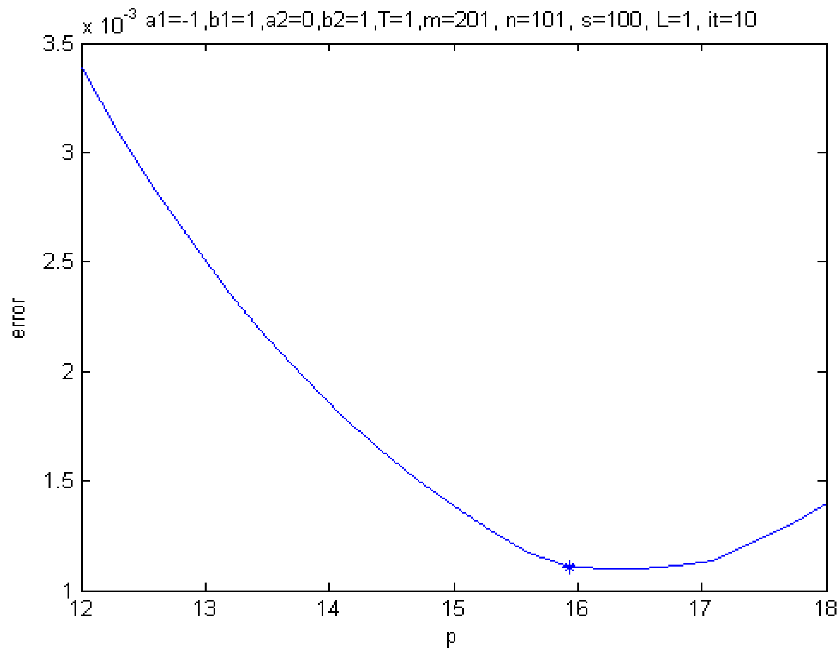

x 10^-3 a1=-1,b1=1,a2=0,b2=1,T=1,m=201, n=101, s=100, L=1, it=10
3.5
3
2.5
2
1.5
1
error
12
13
14
15
16
17
18
p

Figure 3.3.2.D.

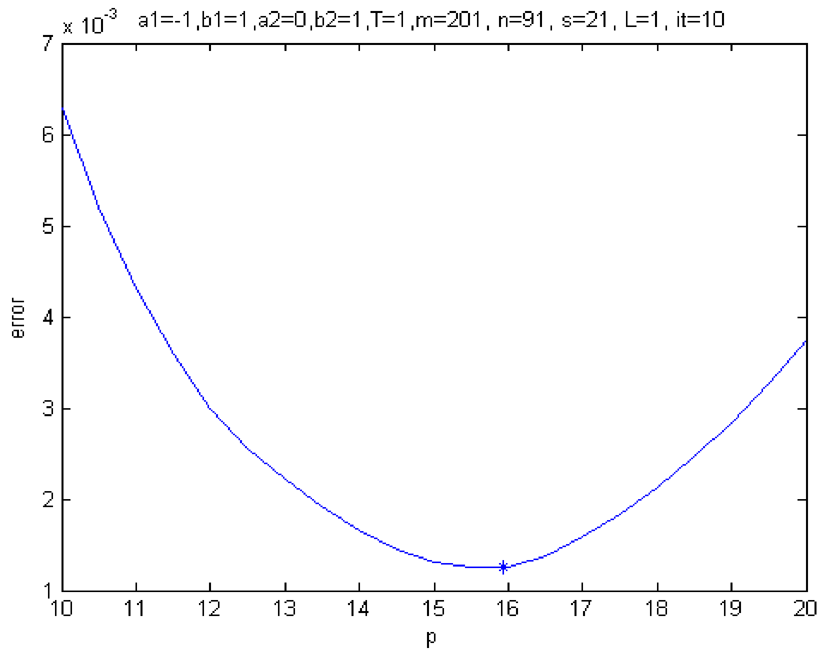

x 10^-3 a1=-1,b1=1,a2=0,b2=1,T=1,m=201, n=91, s=21, L=1, it=10
7
6
5
4
3
2
1
error
10
11
12
13
14
15
16
17
18
19
20
p

Figure 3.3.2.E.

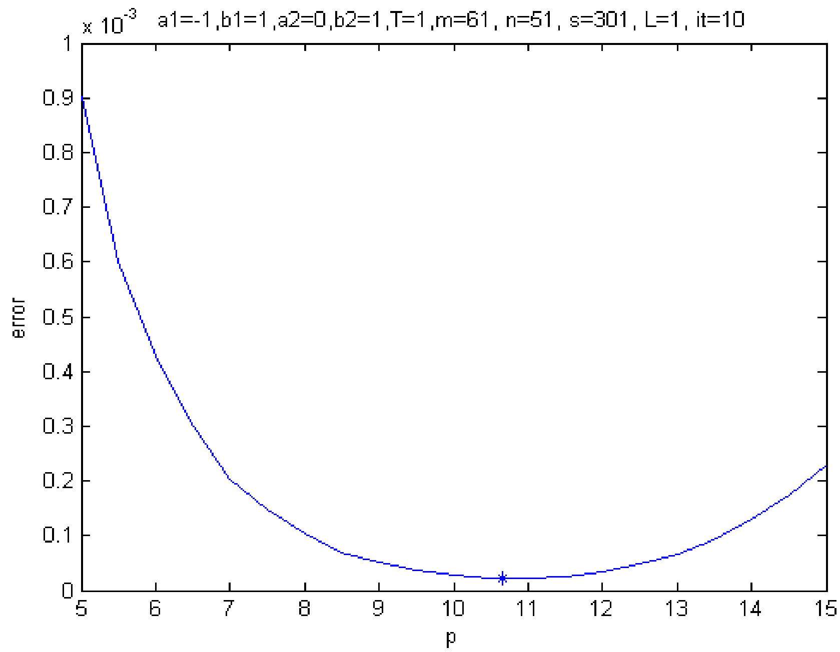

x 10^-3 a1=-1,b1=1,a2=0,b2=1,T=1,m=61, n=51, s=301, L=1, it=10
error
p
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
5
6
7
8
9
10
11
12
13
14
15

Figure 3.3.2.F.

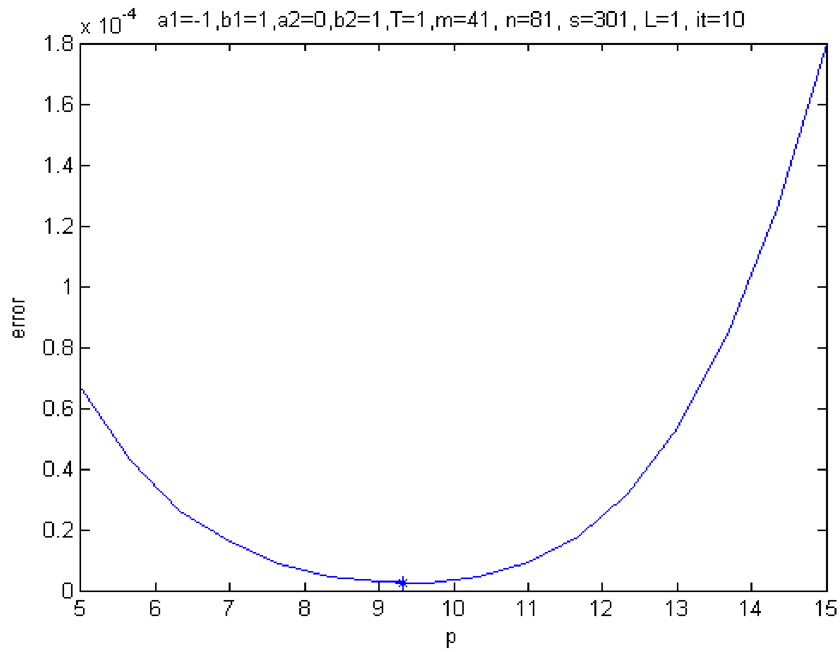

x 10^-4
a1=-1,b1=1,a2=0,b2=1,T=1,m=41, n=81, s=301, L=1, it=10
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
error
5
6
7
8
9
10
11
12
13
14
15
p

Figure 3.3.2.G.

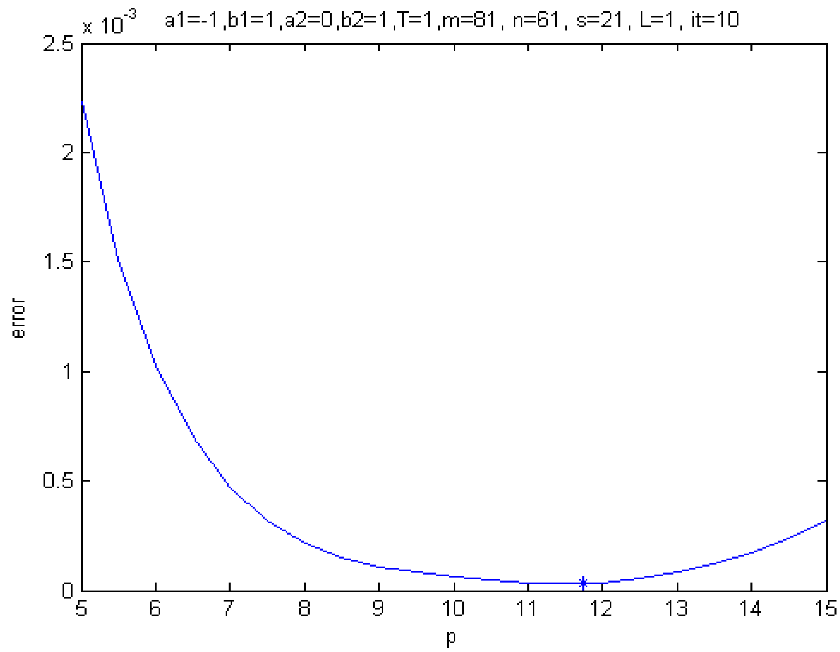

x 10^-3
a1=-1,b1=1,a2=0,b2=1,T=1,m=81, n=61, s=21, L=1, it=10
2.5
2
1.5
1
0.5
0
error
5
6
7
8
9
10
11
12
13
14
15
p
*

Figure 3.3.2.H.

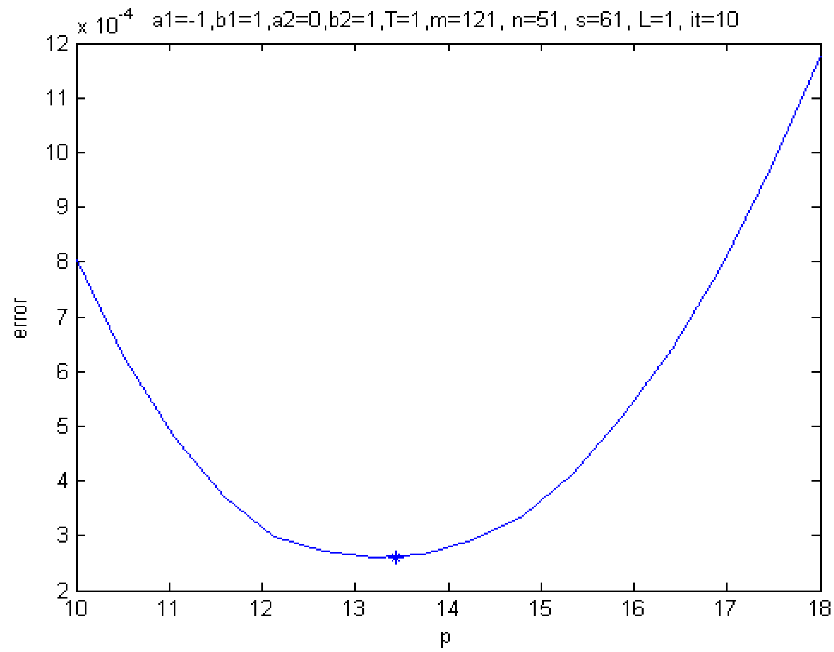

x 10-4 a1=-1,b1=1,a2=0,b2=1,T=1,m=121, n=51, s=61, L=1, it=10
error
p
12
11
10
9
8
7
6
5
4
3
2
10
11
12
13
14
15
16
17
18

Figure 3.3.2.I.

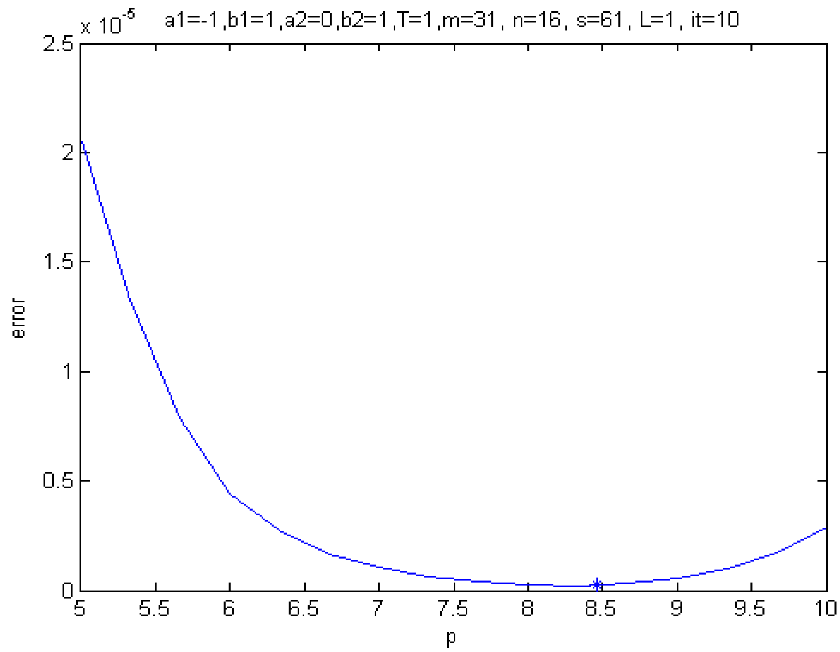

x 10^-5
a1=-1,b1=1,a2=0,b2=1,T=1,m=31, n=16, s=61, L=1, it=10
error
p
2.5
2
1.5
1
0.5
0
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10

Figure 3.3.2.J.

3.3.3 Test 3

Similar as above, we want to test the accuracy of our optimized Ventcell parameters. According to our theoretical results, the optimized parameters depend on the constans C, D in both cases $dt = Cdx$, $dt = Ddy$ and $dt = Cdx^2$, $dt = Ddy^2$. Thus in order to vary C and D, we vary the number of grid points on T, x and y directions to see how the algorithms behave and to see the behavior of the optimized parameter (p, q). The iterations is 16. The test corresponds to both case $dt = Cdx$ and $dt = Cdx^2$.

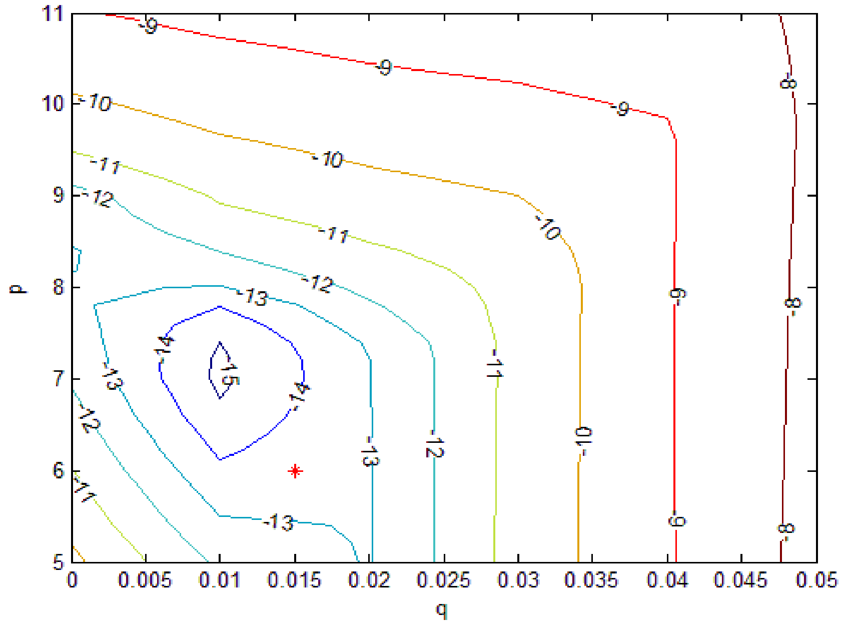
In the first case (Figure 3.3.3.A.), we choose 20 grid points on the x interval, 10 grid points on the y interval and 10 on the time interval.

In the second case (Figure 3.3.3.B.), we choose 60 grid points on the x interval, 30 grid points on the y interval and 300 on the time interval.

In the third case (Figure 3.3.3.C.), we choose 50 grid points on the x interval, 25 grid points on the y interval and 25 on the time interval.

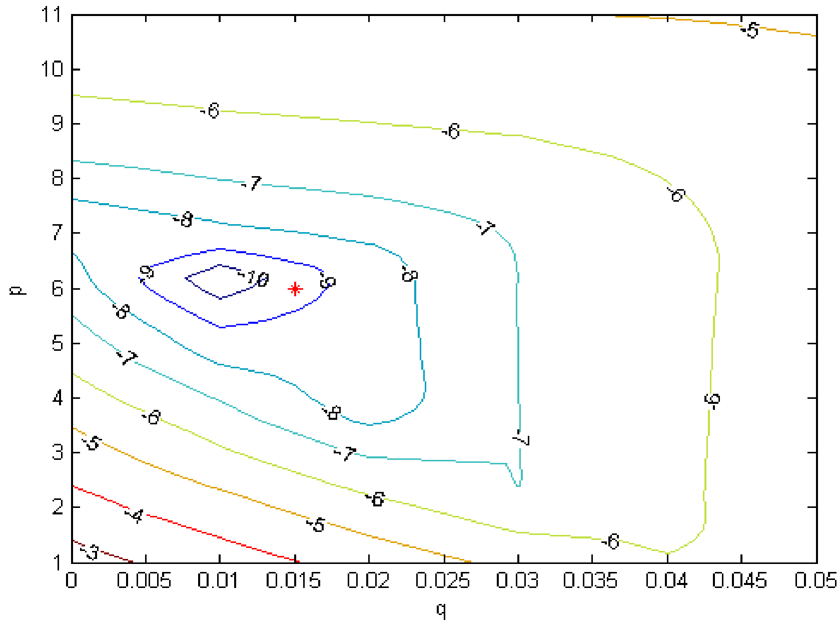
In the forth case (Figure 3.3.3.D.), wwe choose 50 grid points on the x interval, 50 grid points on the y interval and 50 on the time interval.

We can see that the theoretical optimal (p, q) (the '*' on the curve) is quite close to the numerical one.



-9
-9
-9
-8
-8
-8
-10
-10
-10
-10
-11
-11
-11
-11
-12
-12
-12
-12
-13
-13
-13
-13
-14
-14
-15
p
q
11
10
9
8
7
6
5
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05

Figure 3.3.3.A.



11
10
9
8
7
6
5
4
3
2
1
p
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05
q
-5
-6
-6
-6
-6
-7
-7
-8
-8
-9
-10
-9
-8
-8
-7
-8
-7
-6
-6
-5
-7
-6
-6
-5
-4
-5
-3

Figure 3.3.3.B.

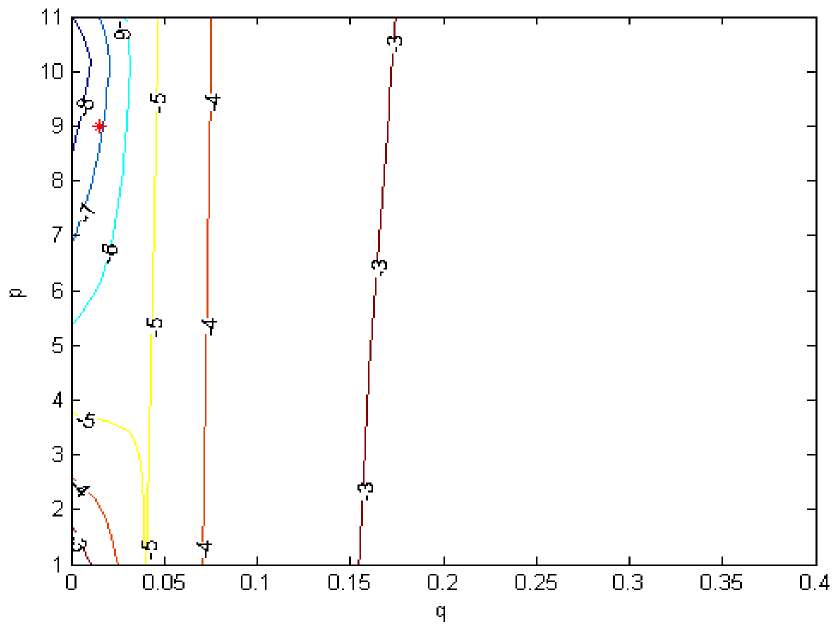


Figure 3.3.3.C.

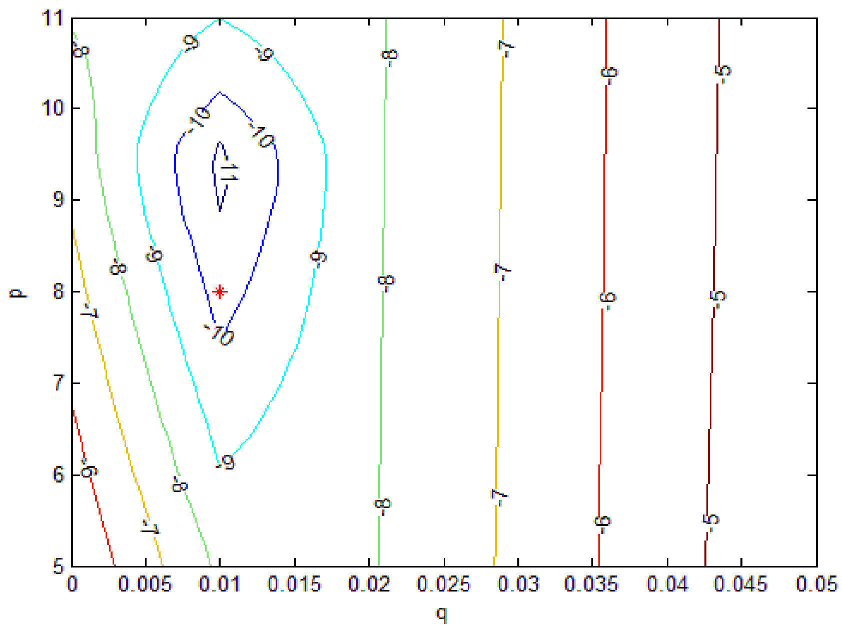

p
q
11
10
9
8
7
6
5
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05
-5
-6
-7
-8
-9
-10
-11

Figure 3.3.3.D.

3.3.4 Test 4

We now want to test our optimized Ventcell parameters, but for the nonoverlapping case. According to our theoretical results, the optimized parameters depend on the constans C, D in both cases $dt = Cdx$, $dt = Ddy$ and $dt = Cdx^2$, $dt = Ddy^2$. Thus in order to vary C and D, we vary the number of grid points on T, x and y directions to see how the algorithms behave and to see the behavior of the optimized parameter (p, q). The test corresponds to both case $dt = dx$ and $dt = dx^2$.

In the first case (Figure 3.3.4.A.), we choose 40 grid points on the x interval, 20 grid points on the y interval and 20 on the time interval.

In the second case (Figure 3.3.4.B.), we choose 40 grid points on the x interval, 20 grid points on the y interval and 201 on the time interval.

We can see that the theoretical optimal (p, q) (the '*' on the curve) is quite close to the numerical one.

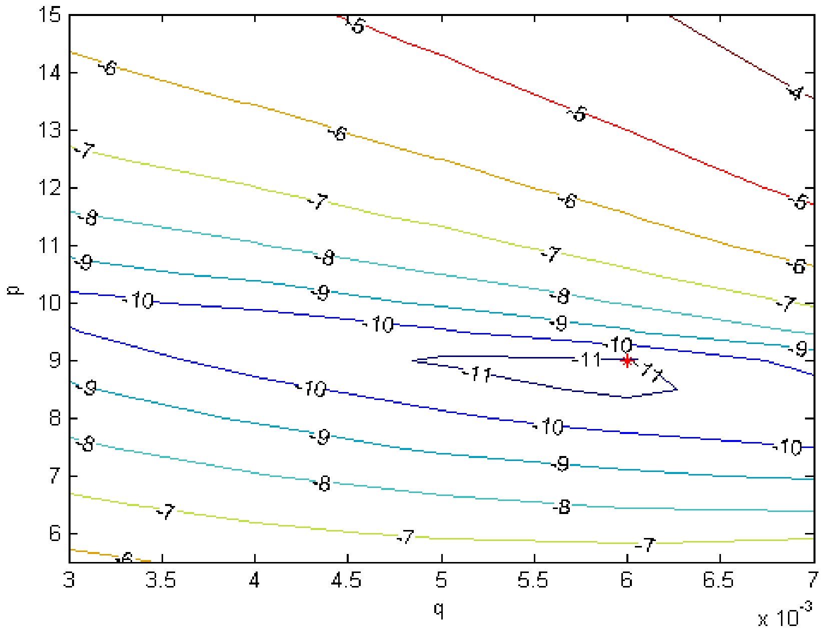


Figure 3.3.4.A.

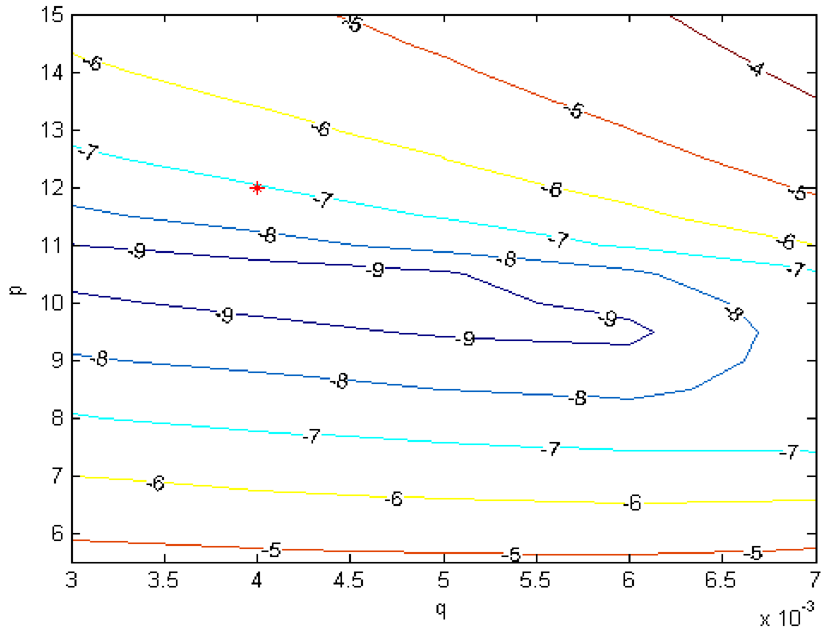


Figure 3.3.4.B.

3.3.5 Test 5

According to our theoretical results, the optimized parameters for the Robin transmission condition have the assymptotic behavior of $Cdx^{-1/4}$. In this test, we want to verify this.

We consider 20 grid points in the x interval, 10 grid points in the y interval and 10 grid points in the time interval, then $dx = dt = 0.1$ and fixed the overlapping length to be 1 grid points. The number of iteration is 15. We repeat this experiment by dividing dx and dt by 2, 3. We plot the practical optimized parameters according to each mesh size and the line $p = dx^{-1/4}$. We can see on Figure 3.3.5.A that the practical optimized line and the line $p = dx^{-1/4}$ are parallel. Which means that the asymptotic analysis predicts very well the behavior of the optimized algorithm.

We consider the same experiment but with 20 grid points in the x interval, 10 grid points in the y interval and 100 on the time interval, then $dt = dx^2 = dy^2 = 0.01$, the overlapping length is again 1 grid points. We repeat this experiment by dividing dx and dt by 2, 3. We plot the practical optimized parameters according to each mesh size and the line $p = dx^{-1/3}$. The asymptotic analysis again predicts very well the behavior of the optimized algorithm in this case.

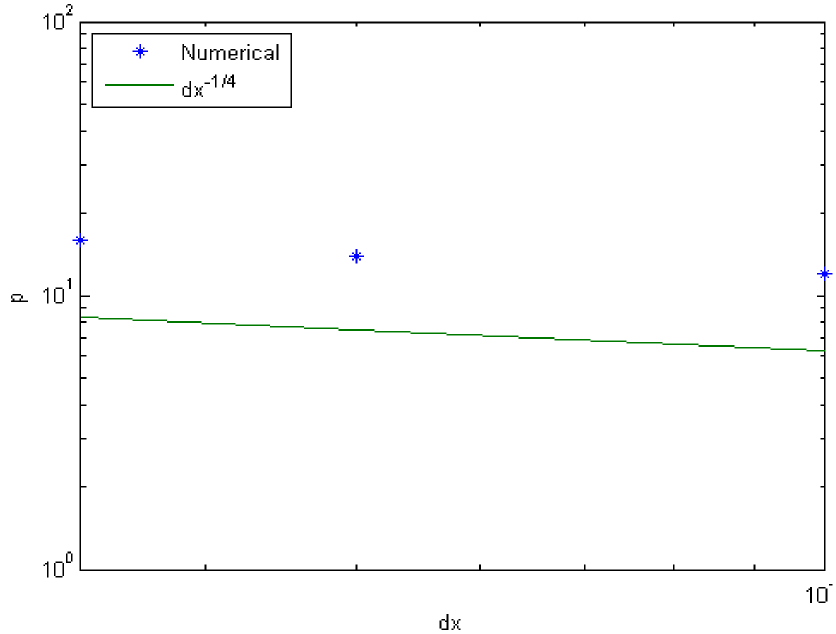


Figure 3.3.5.A

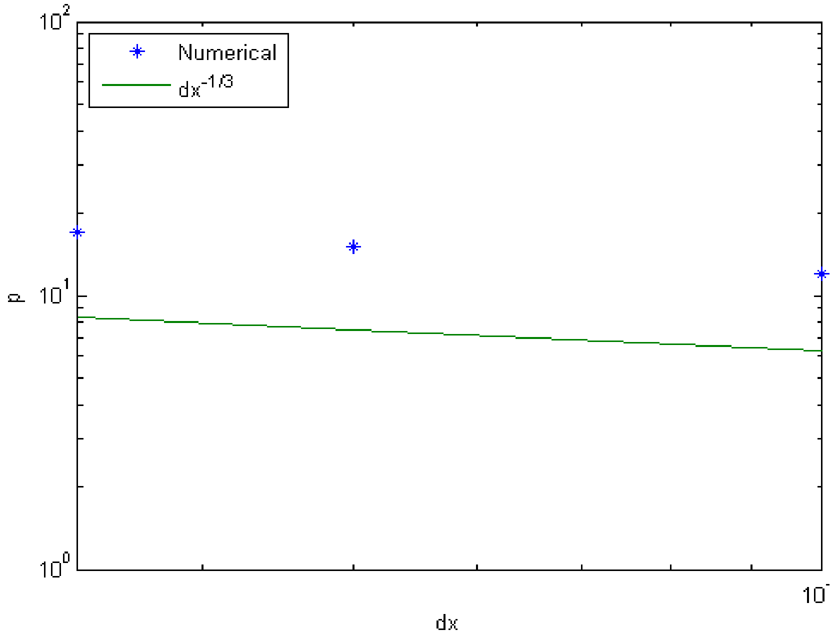


Figure 3.3.5.B

3.3.6 Test 6

According to our theoretical results, the optimized parameters for the Ventcell transmission conditions have the assymptotic behavior of $Cdx^{-/13}$ and $Cdx^{-1/4}$. In this test, we want to verify this.

We consider 100 grid points in the space interval and 200 grid points in the time interval, then $dx = dt = 0.01$ and fixed the overlapping length to be 2 grid points. We repeat this experiment by dividing dx and dt by 2, 3, 4, 5. We plot the practical optimized parameters according to each mesh size and the line $p = dx^{-1/4}$. We can see on Figure 2.3.6A that the practical optimized line and the line $p = dx^{-1/4}$ are parallel. Which means that the asymptotic analysis predicts very well the behavior of the optimized algorithm.

We consider the same experiment but with 10 grid points in the space interval and 200 on the time interval, then $dt = dx^2 = 0.01$, the overlapping length is again 2 grid points. We repeat this experiment by dividing dx and dt by 2, 3, 4, 5. We plot the practical optimized parameters according to each mesh size and the line $p = dx^{-1/3}$. The asymptotic analysis again predicts very well the behavior of the optimized algorithm in this case.

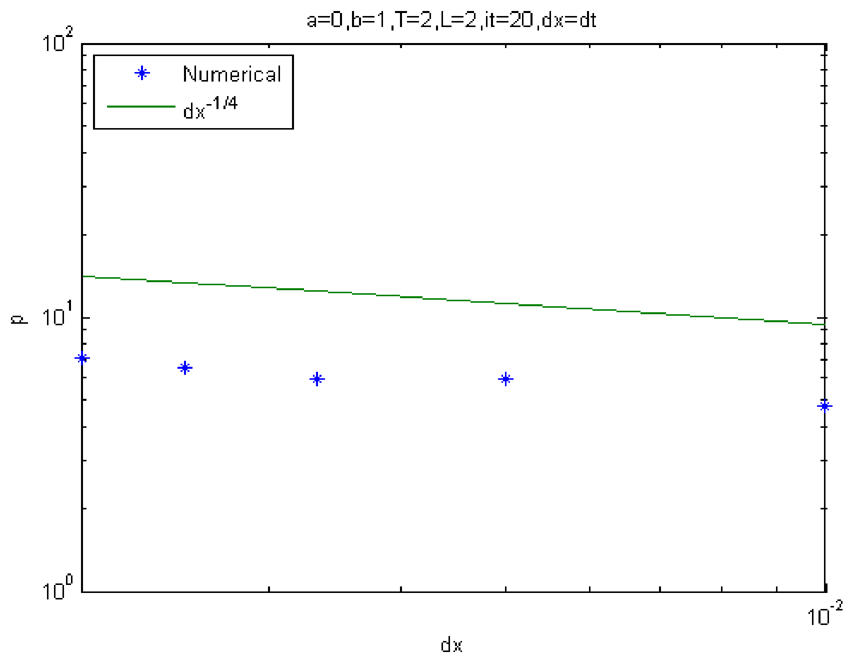


Figure 3.3.6.A

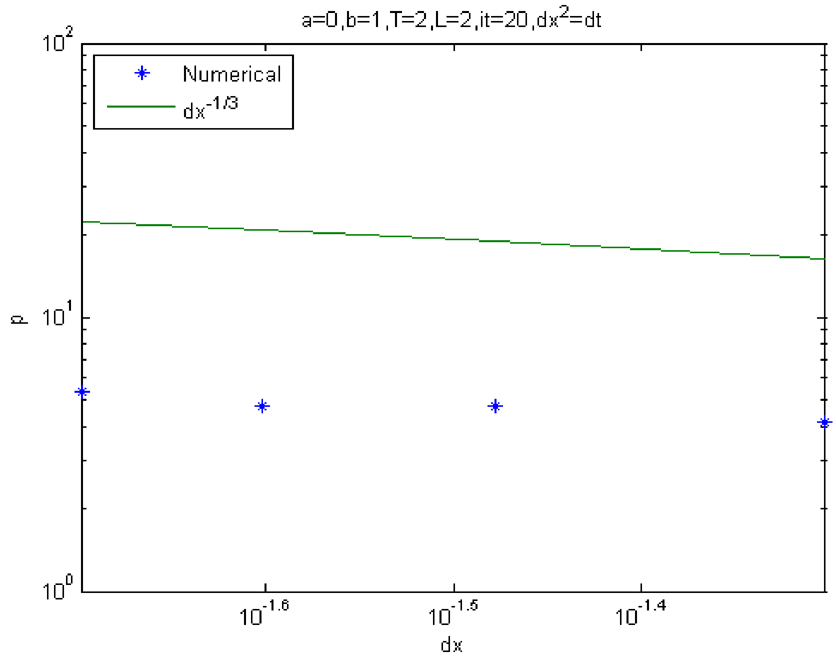

a=0,b=1,T=2,L=2,it=20,dx^2=dt
Numerical
dx^-1/3
p
dx
10^2
10^1
10^0
10^-1.6
10^-1.5
10^-1.4

Figure 3.3.6.B

3.3.7 Test 7

As predicted in our theoretical results, the performance of the optimized Schwarz methods depend on the lengths of the time intervals, we now do some tests on this. We will increase the length of the time intervals, but keep the same dt, and look at the behavior of the methods at each case.

In 2.3.7.A, we consider 20 grid points in the x-interval, 10 grid points in the y-interval, and 10 grid points in the time interval, then $dx = dy = dt = 0.1$ and fixed the overlapping length to be 1 grid points. Then we plot the errors of the methods with respect to the number of iteration. In 2.3.7.B, we increase the time interval from $[0, 1]$ to $[0, 10]$ and choose 1000 grid points on the time interval. In 2.3.7.C, we increase the time interval from $[0, 1]$ to $[0, 20]$ and choose 2000 grid points on the time interval. We can see that the behavior of the methods depends on the length of the time interval.

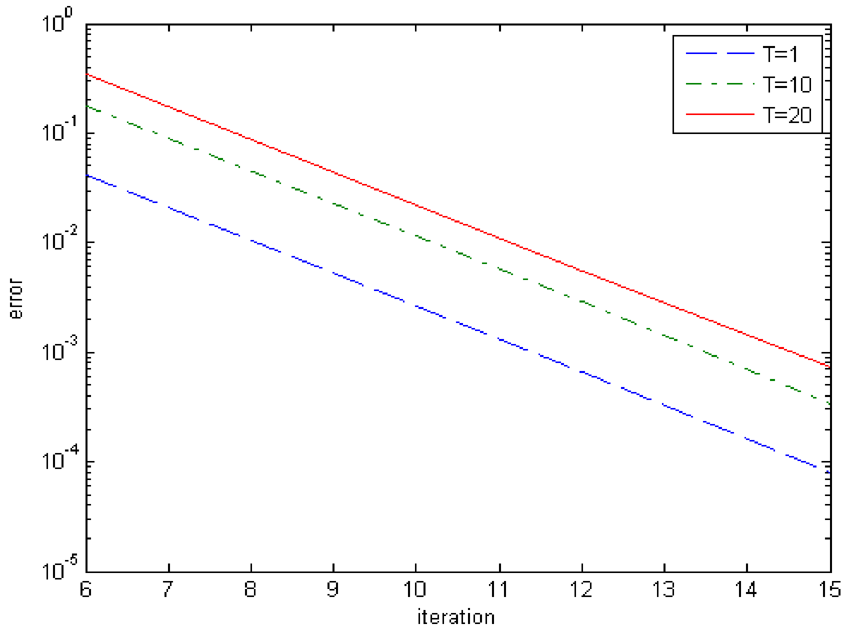


Figure 3.3.7.A

3.3.8 Test 8

In our theoretical results, the performance of the optimized Schwarz methods depend also on the viscosity parameter ν, we now do some tests on this.

In 2.3.8.A, we consider 20 grid points in the x-interval, 10 grid points in the y-interval, and 100 grid points in the time interval, then $dx^2 = dy^2 = dt = 0.01$ and fixed the overlapping length to be 1 grid points. Then we plot the errors of the methods with respect to the number of iteration for the three cases $\nu = 0.1$, $\nu = 1$, $\nu = 10$. We can see that the behavior of the methods depends on ν.

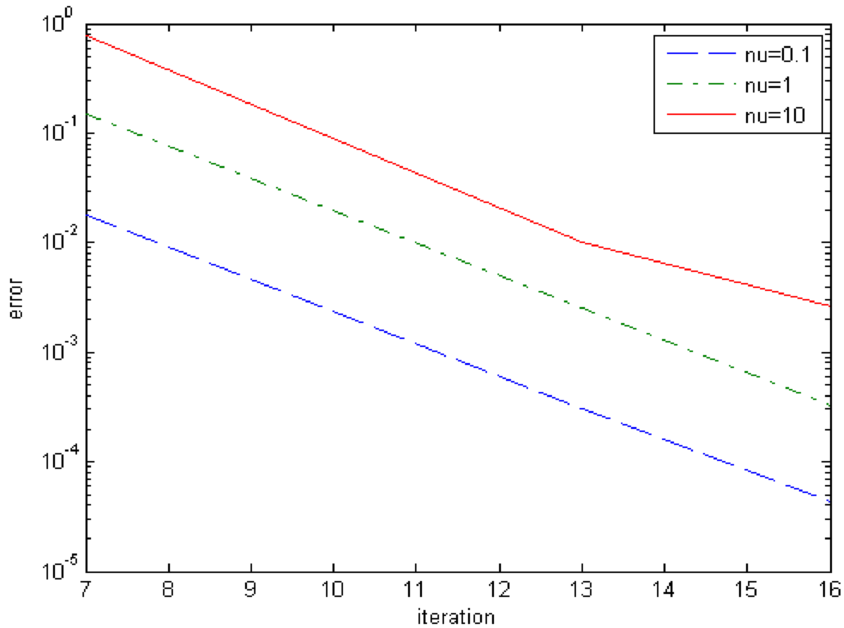


Figure 3.3.8.A

Chapter 4

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